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THE FEASIBILITY OF CINEMA PROJECTION FROM A CONTINUOUSLY MOVING FILM

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ABSTRACT. The paper discusses the optical and mechanical problems involved in the production of screen pictures from a continuously moving film. The film picture may be reflected from a series of mirrors suitably mounted on a drum which is rotated at a definite speed relative to the motion of the film. The various oscillations that may be introduced on the screen image by such projection are discussed, and methods described for minimizing or eliminating them. Calculations are made showing the relation that must exist between the back focal length of the projector lens, the number of mirrors, and the position of stops if definition is to be satisfactory. As compared with the usual type of projection, continuous projection results in a considerable saving of light, much reduced wear and tear of the film, complete freedom from light and dark flickering and consequent eye-strain, and much greater quietness in running.

Introduction

I HAVE been engaged off and on for a considerable time in investigating the feasibility of that principle of cinema projection which consists in projecting stationary and continuously illuminated screen pictures from a continuously moving film. This problem is by no means a new one. It is really many years old, and it might be thought, after reading a few of the patent specifications and papers that have been published professing to solve this matter, that it was a very easy and simple question indeed. My own study of the question, however, has taught me that there are two or three hitherto unsuspected difficulties which must be avoided and can be avoided, if practical success is to be attained.

The usual method of projection, involving the film being jerked forward at least 16 times a second, of course involves far more wear and tear of the film than is desirable, so that the method of continuous feed may be expected to prolong

the useful life of films and reduce to a very great extent the expenses of running them.

Another of the advantages of a system of projecting pictures on to the screen from a continuously moving film consists in, or certainly should consist in, the light projection being continuous, so that the flicker from light to darkness, generally taking place at least 32 times in a second, is abolished altogether. As, however, the question of best light distribution and illumination is a difficult one, demanding very close attention and reasoning, I shall confine myself to general conclusions and deal more explicitly with two difficulties whose avoidance is of crucial importance in the solution of the problem, and which I have not seen mentioned in any patent specification or other literature on the subject.

In Fig. 1, *ABDC* is an outline of one of the pictures on the film. These pictures are supposed to be travelling downwards either by jerks, alternating with stationary periods, as in the existing system, or continuously, as in the newer system, just behind a gate or rectangular opening in a metal plate. This plate defines the limit

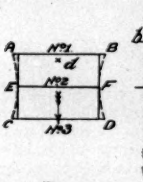


Fig. 1.

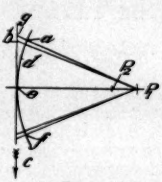


Fig. 2.

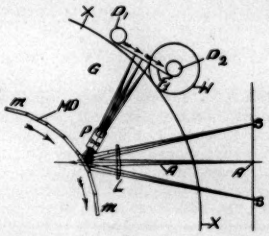


Fig. 3.

of each portion of illuminated film as viewed by the projector lens, whose first and second principal points are at P_1 and P_2 (Fig. 2) respectively. The picture in Fig. 2 is shown subtending an exaggerated angle at P_1 for the sake of exhibiting clearly one of the absolutely essential conditions of correct projection.

In Fig. 3, *m*, *m* are a number (32 in this case) of silvered glass or metal flat mirrors fixed around the circumference of a drum *MD*, which is capable of rotating about an axis placed parallel to the central line *EF* (Fig. 1) of the film picture, and at right angles to the length of the film. This axis is, therefore, perpendicular to the plane of Fig. 2 and the mirror-drum is rotating, at the part presented to the lens, downwards like the film, as shown by the arrows in Fig. 3. These mirrors are lying, as it were, upon a cylindrical surface, that is, they all lie parallel to the drum axis, but each is inclined to its neighbour at an angle equal to $360°/n$, *n* being the number of mirrors; *n* can be any number over 50, which, as I will show, is about the minimum number consistent with tolerably good definition of the whole picture, but 32 only are taken for the purpose of illustration. A mirror-drum may either have its mirrors external for outside reflection, as in Fig. 3, or placed inside the circumference for reflecting inwards, as in Figs. 4 and 5.

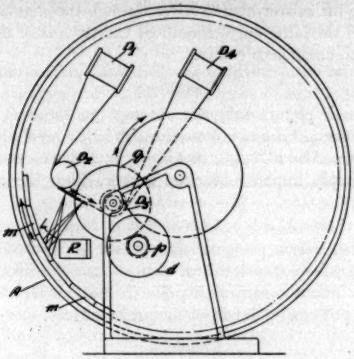


Fig. 4.

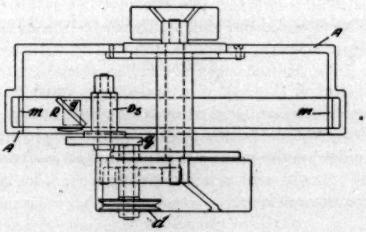


Fig. 5.

Conditions for First-class Cinema Projection

I will now enumerate the most important of the conditions which must be observed if first-class cinema projection is to be secured when the film is in continuous movement. In this enquiry we shall find that we come up against three or four unexpected difficulties in the problem.

Condition 1

The drum of mirrors must be cylindrical, each mirror lying parallel to the axis of the drum, and the axial ray of the projector lens must be reflected by each mirror in a plane perpendicular to the axis round which the drum of mirrors rotates.

Condition 2

The toothed drums which pass the film across the exposure gate must be so geared to the mirror-drum that the former cause the film to travel exactly one

picture pitch while the mirror-drum rotates through the angular interval occupied by one mirror, and the latter angle should be exactly half of the angle subtended by the picture pitch at the lens centre.

Condition 3

Therefore, if each picture subtends an angle $360°/n$ at P_1, the mirror-drum must carry $2n$ mirrors and it must rotate through half a revolution when n pictures have passed the gate. Also it is clear that n may include an extra half-picture since double the half simply implies one extra mirror. Thus the number of mirrors may be odd.

Condition 4

Therefore a fresh mirror presents itself as each picture comes to the gate, so that the reflecting surface of each mirror changes its angle (in the plane of Fig. 2) at half the angular rate at which each point in the film travels downwards with respect to the first principal point P_1 of the projector lens.

Condition 5

Therefore the distance eP_1 (Fig. 2) must be exactly the radius of a circle around which n or $n + \frac{1}{2}$ picture intervals can be exactly wrapped. The picture interval is generally supposed to be ·75 inch, but in practice varies between ·737 and ·748 inch in different films.

Condition 6

If a focussing lens L (Fig. 3) of a focal length equal to the distance to the screen is to be used, then eP_1 (Fig. 2) must be exactly the principal focal length of the projector lens, so that the rays from points in the picture shall emerge from the lens and impinge on the mirrors as pencils of *parallel* rays. In such a case the projector lens is really used as a collimating lens. The reason for this latter condition will be explained later.

Condition 7

If we use our new type of projector lens (described later and illustrated in Fig. 13), which enables the image of the arc light to be focussed outside of the projector lens and upon the mirrors, the focussing lens L is no longer necessary and the distance eP_1 has to be equal to the back conjugate focal distance of the projector lens. The rays of pencils then leave the lens in a slightly convergent state so as to focus upon the screen. This is the condition which it is most desirable to adopt.

Condition 8

Since the angular velocity of the mirror-drum, and therefore the angular rotation of each mirror, is uniform when the linear travel of the picture is uniform, it is quite clear that the movement downwards of any point d (Figs. 1 and 2) with respect to P_1 must be uniform if the mirrors are to project the pencil of light from d to a constant position or fixed point on the distant screen. This means that the point d or any other point must be virtually made to travel at a uniform velocity around a cylindrical surface *adef* (Fig. 2), the radius of which is equal to eP_1. If so,

then it may be asked, why should the film not be made to run round against a cylindrical gate of radius eP_1? The reason is that, if the straight film *bec* be wrapped around the cylindrical surface so that the upper straight border (*AB* in Fig. 1) of the picture at *b* wraps round to *a* and the lower border to *f*, the two horizontal borders *AB* and *CD* will be lengthened out as viewed from P_1 so that the vertical borders of the picture *AC* and *BD* will appear to the point P_1 to take a strongly bowed shape, as shown dotted in Fig. 1. This distortion cannot be tolerated, for although points like *d* anywhere on the vertical median line of the picture will move correctly and be correctly placed on the screen, points near the right and left margins of the picture cannot be well defined as they will vibrate along short horizontal straight lines, of lengths equal to the versine of the curved ends of the picture, during the passage of each picture through the gate.

The avoidance of these defects and considerations of the projector lens performance demand that the flatness of the pictures must on no account be substantially departed from, provided that the projector lens gives a rectilinear and flat field; we therefore come up against the following difficulty. If points in the picture are moving downwards in the plane *bc* (Fig. 2), it is clear that they are moving with respect to P_1 with a constant *tangential* velocity, that is, the tangent be/eP_1 changes uniformly. This, however, will not suffice for accurate reflection by the mirrors to fixed points on the screen. For such accurate reflection the points of the picture must move or seem to move downwards with a constant angular motion, *i.e.*, their movement in *radian* measure must be kept constant, while at the same time the picture must appear to the lens flat and undistorted.

Let it be supposed that the film *eb* is wrapped round the cylindrical surface *eda* so that the point *b*, representing the higher border of the picture, wraps round to *a*. Draw P_1b and P_1a and produce P_1a to *g*. Thus we have the tangent measure eb/eP_1 converted to the equal circular or radian measure ea/eP_1, that is, the tangent of the angle bP_1e is equal to the radian measure of the angle aP_1e and unfortunately these angles are by no means equal.

If any point *d* travels or seems to travel uniformly along the arc *ea*, it is exactly the same to the lens as if, with regard to position, it had been made to travel (at an accelerating rate) in the plane *eb* up to the point *g*, so that it appears to be at *g* at the same instant that it would arrive by circular travel at *a*. If the half-picture angle bP_1e be greater than 5° to 6°, it can be shown that the difference of actual angles corresponding to two tangent and radian measures that are equal becomes very serious. This has been largely responsible for previous attempts at such projection from continuously moving films being unsuccessful. For the sake of economizing the number of mirrors, the focal lengths of the projector lenses used have been kept far too small.

Table I shows how the inconsistency of angle for equal tangents and radian measures mounts up in the ratio of the cubes of the tangents, the focal distance eP_1 being 1·4135 inches, for 12 pictures per 360° and half a revolution of the mirror-drum (carrying 24 mirrors). Thus each picture pitch of ·74 inch represents an angle of 30° at P_1.

Table I.

Back focal length = 1·4135 inches. Pitch = ·74 inch.

A	B	C		D
Travel of point *d* on film from the median line inch	Radian measure or tangent of angle bP_1e	Shortage of angles estimated by tangent measure as compared with equal radian measure, in seconds and in minutes		Distortion of position, being shortage divided by travel (both in seconds)
		secs.	mins.	
·06	·0424474	− 5	0	− ·00057
·12	·0848948	− 42	− ·7	− ·002424
·18	·1273422	− 140	− 2·3	− ·00543
·24	·1697897	− 331	− 5·5	− ·009696
·30	·2122370	− 638	− 10·6	− ·01525
·36	·2546845	− 1092	− 18·0	− ·02215
·375	·2652991	− 1226	− 20·4	− ·02400

Column C gives the radial oscillation occurring during the travel of any point in the film picture from the edge of the gate to the centre or *vice versa*, and it is clear that during the further travel from the centre of the gate to the other edge this oscillation will be repeated, but in the opposite direction, so that the total oscillation taking place is double that given in column C. It will be noticed that the proportionate negative distortion in column D increases as the square of the travel in column A, while the linear or absolute amount in column C increases as the cube of the travel and therefore follows the same law as the distortion given by a lens. Therefore a simple or compound cylindrical lens, which we might call a converter lens, could be designed and placed with its lens axis along the median line of the gate and at right angles to the direction of travel of the film, so as to give positive distortions sufficient to counteract the above negative distortions and make. for instance, the point *b* appear to be at *g* in line with *a* as seen from P_1. It will be seen from Table I that any travel less than ·135 inch (for an angle of about 5°·5) does not involve angular discrepancies greater than 1 minute of arc, which is about as much as can be tolerated in really first-class cinema projection. This would mean that the whole picture pitch of ·74 inch would subtend at the projector lens (back focal length 3·875 inches) an angle of nearly 11° and there would be 33 pictures to a complete circuit of 360° and 66 mirrors round the mirror-drum. The cylindrical corrector or converter lens would no longer be either necessary or desirable, especially if the angle of view were further reduced and the number of mirrors correspondingly increased, which conditions might be required for a cinema hall of more than average length.

It will be seen that the discrepancies with which we have been dealing must inevitably cause the images of all points within the picture to vibrate up and down on the screen as the corresponding points in the film pass downwards across the gate opening. I call this the minor angular oscillation. This oscillation can, however, be reduced in effect to about three-eighths by a compromise in the shape of slightly reducing the back focal length of the projector lens by a proportionate fraction equal to the figure given in column D. For instance, if the reduction in

the present back focal length of 1·4135 inches is ·024 *F*, or ·035 inch, the oscillation will be reduced to 37·5 per cent., which is the best possible compromise. This will be illustrated later.

Condition 9

Let us now consider another condition still more important and necessary than the last one if really first-class cinema projection is to be attained. Fig. 6 represents a case of projection by the lens *L* on to the picture screen *AB*. The picture screen plane therefore passes through *AB* and is perpendicular to the diagram. *AB* is the height of the projected picture and is parallel to the length of the film and to its direction of movement and perpendicular to the axis about which the mirrors rotate. When any film picture is centrally placed in the gate,

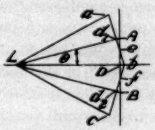


Fig. 6.

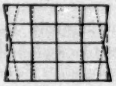


Fig. 7.

all or the great bulk of the light from that picture and issuing from the projector lens is reflected from one mirror and goes to form the screen picture *AB*. If any light is reflected from the two neighbouring mirrors, it goes to form two other repetition pictures lying above and below *AB*, but these are supposed to be cut off by a suitable stop placed a few feet from the projector. When, however, the gate is occupied by half of one picture and half of the next picture, and the line of demarcation between them lies across the centre of the gate, two reflecting mirrors are concerned which are so placed that their junction lies on or near the lens axis, and therefore one mirror receives and reflects half of the light pencils emerging from the projector lens, and the other mirror the other half. Under this condition, while the screen picture *AB* appears stationary and complete, it really corresponds not to one film picture, but to the upper half of one film picture joined on to the lower half of the neighbouring picture. But the illuminated gate opening, unlike the film pictures, is immovable and fixed, and therefore the images of it thrown by the two mirrors on to the screen are displaced up and down on the screen. In the case considered, one mirror is throwing the projection of the lens axis (normally *Lb*) up in the direction *LA* and the other mirror is throwing it down in the direction *LB*. One mirror is causing a correct and undistorted image of the gate and its contents to be formed in the plane *ab* perpendicular to the reflected lens axis *LA* and the other mirror causes a similar undistorted image to be formed in the plane *bc* perpendicular to the reflected lens axis *LB*.

The skewed picture plane *ab* cuts through the plane of the screen *AB* at *e*, while *bc* cuts through the plane of *AB* at *f*, and it is clear that *e* and *f* are half-way between the centre and upper and lower margins of the screen picture *AB*. Now it must be realized that the image of the top margin of the gate, which is at the same time the central line of one film picture, is a straight line perpendicular to the diagram which passes through the point *b*, while the image of the central line of the gate, which is, at the same time, the dividing line between two adjacent film

pictures, is also a line perpendicular to the diagram which passes through the point d_1 and, as thrown by the other mirror, also through d_2.

Therefore the top margin of the screen picture at A is lengthened out beyond the normal in the ratio LA/Ld_1 and the bottom margin in the similar ratio LB/Ld_2. At the same time the central horizontal line of the screen picture at D is shortened in the ratio LD/Lb.

Therefore the upright or end margins and other vertical lines of the screen picture appear as in Fig. 7, where in the top half the vertical lines are straight but converging downwards and in the lower half straight but converging upwards. The waist of the picture is compressed while the top and bottom margins are enlarged. Hence points at or towards each end of the waist of the picture have oscillated horizontally inwards while points at the corners have oscillated horizontally outwards, and if the length of this oscillation subtends a greater angle than 2 or 3 minutes as viewed from the lens centre, first-class definition of the ends of the picture cannot be attained. A formula for the angular value of this oscillation is therefore required, and for the purpose of working it out we may legitimately take as our quantities the dimensions of the film picture in relation to the back focal length of the projector lens, which relation is the same as that which the dimensions of the projected picture bear to this projection distance.

In Fig. 6 let AB represent the picture on the film and θ be the angle subtended by half the vertical height of the picture at L (the lens centre), LD being the back focal length (F) of the projector lens. Then $AD = F \tan \theta$.

Also the angle $\quad Aed_1 = \theta$ and $Ae = \dfrac{LD \tan \theta}{2}$.

It is clear that $\quad Ad_1 = \dfrac{LD \tan \theta}{2} \cdot \tan \theta,$

and therefore $\quad \dfrac{Ad_1}{LD} = \dfrac{\tan^2 \theta}{2}.$

If H is half the horizontal length of the picture, H/LD or H/F is the angle subtended by it, and then the angular value of the magnification or diminution or displacement of points at the corners and the ends of the waist is expressed by $\dfrac{\tan^2 \theta}{2} \cdot \dfrac{H}{F}$, in which H is usually ·50 inch. If AD, half the height of the picture AB, is expressed by V, then $\tan^2 \theta$ becomes V^2/F^2 and the expression becomes $\dfrac{V^2 H}{2F^3}$,

where $\quad F =$ the back focal length of the lens,

$\quad V =$ half the height of the film picture,

and $\quad H =$ half the width of the film picture.

This expresses the angular value of the oscillation as viewed from the lens centre and Table II gives the angular oscillation for four different focal lengths. We call this the major angular oscillation.

Table II.

Back focal length ins.	Major angular oscillation mins.	Minor angular oscillation (see Condition 8) mins.
2·00	14·6	6·6
3·016	4·3	2·0
3·96	1·88	·90
4·86	1·02	·46

Hence any focal length below 3·016 inches is incapable of giving first-rate or even second-rate definition at the ends and corners of the projected pictures, and in order that the definition shall be really first-class, the focal length must not be less than 3·5 inches. Luckily, however, the oscillation decreases in proportion to the distance of the vertical lines from the centre of the picture, as shown in Fig. 7.

We may now realize that the oscillation of the ends of the picture brought about by the above explained cause, together with the vertical minor oscillation (dealt with in Condition 8) of all points in the picture caused by the differences of angle between equal radian and tangent measures, have been the chief causes why so many attempts at cinema projection by rotating mirrors have been practical failures. As far as we know, too short focal lengths for the projector lens, combined with the necessary corollary of too few mirrors in the mirror-drum, have hitherto been employed. For a back focal length of 3·6 inches a drum carrying 60 mirrors is necessary. It will be observed in the above table that the major oscillation of end points is really more than twice that of the minor oscillation affecting all points, dealt with in Condition 8, and we consider that no practical way of overcoming this major defect can be devised. Thus it becomes useless to employ cylindrical converter lenses, etc., such as we have before suggested, for overcoming the difference between radian and tangent measures, since that difference becomes negligible at the focal length (say 3·54 inches) which just permits of the major oscillation of end points not exceeding 3 minutes of arc. It is also obvious that the further vertical opening out of the gate so as to include two picture pitches in order to get images of two separate pictures superimposed on the screen, as has been attempted, must necessarily cause a further disastrous increase of the two oscillations. The major oscillation would be increased fourfold and the minor oscillation eightfold. The effect upon the major oscillation is shown in Fig. 7 by the dotted continuations of each picture where it overlaps the other half-picture.

We must now revert to our Condition 6 which has special reference to projection by ordinary lenses and explain how necessary it is, when employing any ordinary type of projector lens with central stop together with an external focussing lens L (of Figs. 3, 4, 8, 9, 10, and 11), that the rays radiating from each point of the image and emerging from the projector lens, whose principal points are P_1 and P_2, shall emerge as pencils of parallel rays before being reflected by the mirrors.

In Fig. 10, P_2 is the second or outer principal point or stop image of the same lens as is used in Fig. 2. Let it be supposed that the central point C of a picture is travelling downwards and is the source of a diverging pencil of rays which fill the aperture (say $\frac{1}{2}$ inch) of the lens and then emerge from the lens as a parallel

beam. When C first shows within the gate, the central or principal ray of the emerging pencil takes the direction P_2r_1 (Fig. 10) and the mirror lie is m_1m_1; when C gets to the centre of the gate, the principal ray of the emergent pencil takes the direction P_2r and the mirror lie is mm; and when C gets to the bottom of the gate the principal ray of the emergent pencil takes the direction P_2r_2 and the mirror lie becomes m_2m_2. Now if the reflecting surface of the mirror can be made to pass through P_2, the angular rotation of the mirror while it is in use will reflect all these principal rays P_2r_1, P_2r, and P_2r_2 along one line P_2Q and they can then be all focussed upon the screen by a slight focussing movement of the projector lens itself. But, when ordinary projecting lenses are employed, the mirrors have to be at a considerable distance outside of P_2 and they necessarily therefore reflect all the above principal rays in directions parallel to one another, but considerably separated as if originating from different apertures p_1, p, and p_2 and the further off the mirrors are from the projector lens the greater this separation and the apparent

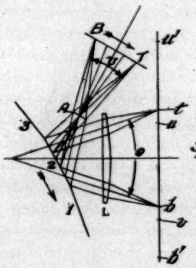


Fig. 8.

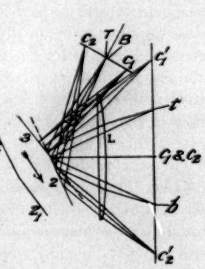


Fig. 9.

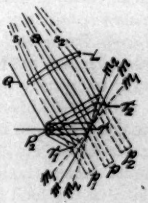


Fig. 10.

movement of the lens pupil become. In this case the pencils seem to originate from a rapidly oscillating lens aperture p_1, p, p_2.

If, under these conditions, the projector lens were focussed so that the rays constituting each of the pencils s_1, s, and s_2 were caused to converge slightly so as to focus upon the screen, this point would, during the passage of the point C down the gate, obviously move on the screen through the distance p_1p_2 and the definition would be ruined. What should be a fixed point would be rendered as a vertical straight line of oscillation. But if the individual pencils s_1, s, and s_2 are collimated by the projector lens and therefore each consist of parallel rays, they are in the same condition of parallelism as the central or principal rays r_1s_1, rs, and r_2s_2 are in to one another. Therefore a simple positive focussing lens L, of a focal length equal to the distance to the screen (which may be 50 feet or more), placed just clear of the mirror-drum and squared and centered upon the reflected normal principal rays rs, is all that is necessary in order to converge all the separate and parallel pencils emanating from each point of the film picture on to exactly the same point

on the screen. In this way the apparent shifting of the lens aperture or pupil between the positions p_1, p, and p_2 is completely nullified in its effects.

It will be observed that, if the focussing lens L were not there, the effect would be just the same as if the projecting lens, together with the film behind it, were oscillating up and down parallel to its optical axis between the positions p_1 and p_2. A vertical oscillation of 0·18 inch produced in this way would correspond to 1 minute of arc on a screen placed 50 feet from the lens, so that ·90 inch of oscillation would cause 5 minutes blurring. Fig. 10 is meant to give a general idea of the lateral parallel transference of the reflected principal rays emanating from the central point of the picture and is considerably exaggerated. A subsidiary cause of the same lateral parallel movement is left out, for it is not strictly true that the reflecting surface of the mirror during its transit always passes through the one point r. It will be easily seen that there is also a slight rise and fall of the surface of the mirror as it passes and rotates in front of the lens, but this rise and fall does not affect the angular inclination of the mirror to the projector lens axis, but only slightly modifies the amount of lateral parallel transference p_1p_2.

The lens L, which we call the focussing lens, is of such a long focal length relative to its aperture that there is no necessity for its being compound or achromatic.

Condition 10

Since there are two forms of mirror-drums available, the one illustrated in Fig. 3, in which the mirrors are fixed on the outer circumference of the drum so as to reflect outwards, and the other in Fig. 4, in which the mirrors are fixed on the inner circumference so as to reflect inwards, we must now consider the advantages and disadvantages of each form of mirror-drum, and what conditions are necessary in each case to secure the best results.

We will first consider the outward reflecting mirror-drum. Fig. 8 shows an ordinary projector lens whose aperture is A, as defined by a stop somewhere within its mounting. BT is the gate, and when a film picture is accurately placed in the gate it is upside down, its bottom being at B. The rays diverging from T, after emerging as a parallel pencil from A, impinge upon the top or rear margin of the mirror 2, and the rays diverging from B are similarly reflected as a parallel pencil from the lower or forward end of mirror 2, and both the film and the mirrors are moving in the direction of the arrows, left down to right.

Let it be supposed that the picture TB has then moved on by, say, one-eighth part of its pitch, and at the same time the mirror 2 has moved about one-eighth of its width downwards and therefore the next mirror 3 has obtruded one-eighth of its width into the pencil of light from T. Then, although the picture tb on the screen has kept stationary as to its details, the illumination area (corresponding to the gate and reflected from 2) has moved downwards one-eighth and extends from u to v, and the strip tu, being the top of the screen picture, is left dark while the illuminated strip bv falls below the proper screen picture and is an image of the top of the next picture (now beginning to show at B) which is missing at tu. Also the light that is first reflected from the oncoming mirror 3 comes from the

lower edge *T* of the gate and forms a portion of the screen picture just below the highest illuminated edge *u*, and as this light is reflected by a mirror that is tilted back relatively by half the picture angle, the image on the screen thrown by 3 is thrown up to *u'*, the interval *uu'* being the height of a complete picture; this light is useless, and at the expense of the illumination of the corresponding part *tu* of the proper picture. Moreover, if the distribution of the light pencils is traced out at the phase when the division between the two pictures lies across the centre of the gate and the junction between the two mirrors lies on the lens axis, and both mirrors are equally operating, as in Fig. 9, it will be seen, in the case of the ordinary projector lens *L* of Figs. 8 and 9, that the proper screen picture will shade off in brightness from the edges down to the centre. The centre will show a more or less dark band and the top and bottom margins *t* and *b* will receive only half light, while the supernumerary false half-pictures $c_1't$ and bc_2' (Fig. 9) projected above and below will be brighter than the picture proper.

This drawback can be overcome in two ways:

(1) One may substitute another pair of mirrors like 2_1 and 3_1 (Fig. 9) also inclined at half the picture angle with one another, but an angle of less than 180° instead of greater, as in Fig. 4. This means that the drum of mirrors must be hollow with its mirrors facing inwards. Then the bright lower half-picture c_1t will be transferred down to c_1b and the bright upper half-picture bc_2' will be transferred to c_2t, and the whole be made complete from *t* to *b*.

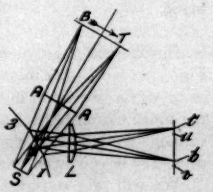


Fig. 11.

(2) One may substitute another lens of the type shown in Fig. 11. The conditions of illumination when the gate is occupied by two half-pictures will then be reversed, for the light will increase from half light at the top and bottom of the picture to full, or nearly full, light at the centre, while the supernumerary false half-pictures will diminish further in brightness according to the distance from the upper and lower margins of the picture proper.

Thus this transfer can be effected either by employing a hollow mirror-drum with its mirrors reflecting inwards, as in Figs. 4 and 9, and using a projector lens of ordinary form with its stop or pupil within its mount, as in Figs. 8 and 9, or else by retaining the outward reflecting drum and employing a projector lens, as in Fig. 11, with its stop or pupil point virtually arranged to be external to the lens and lying either on or even behind the mirrors, as in Fig. 11. Moreover, this latter plan enables us to dispense altogether with the focussing lens shown in Figs. 3, 8, 9, and 11, because with the pupil or image of the electric arc focussed upon or nearly upon the mirrors, it no longer oscillates up and down as described and illustrated in Fig. 10.

As the internally reflecting mirror-drum obviously involves greater difficulties in construction and more complication optically than the externally reflecting mirror-drum, it is decidedly advantageous to adopt the latter, and Fig. 11 illustrates what happens when the next oncoming mirror 3 is beginning to receive light, which

in this case originates not from *T*, but from *B*, where the top of the next film picture is just showing itself. This light is partly reflected by mirror 2 to *bv* just below the screen picture, but with rapidly diminishing brightness, while the oncoming mirror 3 is rapidly appropriating more and more of the light from the top of the next film picture appearing at *B*. Owing to the angular back tilt of mirror 3, the image of the strip is thrown up to *tu* where it completes the whole screen picture *tb* and this transfer amounts in angle to a whole picture pitch.

It can be shown that the best general results in illumination will follow when the mirrors are arranged to be narrow enough to be slightly overlapped by the axial pencil of parallel rays emerging from the lens. Each mirror should be of such a central width as to intercept about $\frac{3}{4}$ or $\frac{8}{10}$ of the cross-section of the axial beam emerging from the lens, supposing the latter to have an effective aperture *F*/5. A good practical rule is that the central width of the mirrors should be one-hundredth part of the number of mirrors on the drum. If the latter is 64, then the mirrors should be ·64 inch wide. This will imply that when any film picture is central in the gate and its corresponding mirror is central, there will be formed immediately above the screen picture a more or less narrow repeated strip of the bottom of the same picture and just below the screen picture a more or less narrow repeated strip of the top of the same picture. These spurious images, however, can be screened off, partly by interposing a rectangular opening a few feet in front of the projecting lens and also by framing in the white picture screen by broad strips of comb-moulded wood or other material of a section such as is shown in Fig. 12, the strips being painted dead black. Owing to the very oblique angle of incidence of the light upon the sides of the teeth-like mouldings, the light reflected back to the spectators will only be about one-fifth of what it would be from a blackened plane surface.

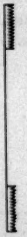


Fig. 12.

As regards the projector lens, while a real stop cannot be placed at *S* (Fig. 11) behind the mirrors, yet a virtual stop can be, for it amounts to practically the same thing if the image of the electric arc light is formed at *S*, either on or just behind the mirrors, and is so arranged as to have a diameter equal to about *F*/5 or *F*/6 (*F* being the E.F.L. of the lens), while the aperture of the lens or lenses themselves is *F*/4·5 or *F*/3·5 so as to allow of the outward spread of the pencils from the virtual stop to the lens.

Cooke Lantern Projector Lens

The desideratum that the lens should give first-class definition of the picture on the screen depends upon the lens being properly designed to suit the case, and I have worked out a form of Cooke Lantern Projector Lens, consisting of three simple and separated lenses, to fulfil the above special requirements. Fig. 13 shows the form of this lens, consisting of a negative lens *N* placed between two positive lenses P_1 and P_2, the power of the negative lens being roughly 86 or more per cent. of the sum of the powers of the two positive lenses. The entrance stop is at S_1, which appears after refraction by the lens to be $1\frac{1}{2}$ times enlarged at S_2. Conversely, the condensers first form the image of the arc at S_2 so that after passage through

the lens the arc is focussed at S_1, this real image lying on or even slightly behind the mirrors as above described.

Fig. 14 is a rough illustrative sketch to about one-fifth scale of a machine in which the film *F* is pulled regularly over the following drum D_1 and through the gate by means of the drum D_2, which is provided with the usual two rows of cogs for engagement with the two rows of apertures punched in the film, there being four pairs of teeth per picture pitch. The circumference of this drum is shown to correspond to 4 pictures, so that 8 revolutions pass 32 pictures and the mirror-drum *MD* is turned through one half-revolution, there being 64 mirrors mounted on it. This is done most simply by the spur-wheel *W*, gearing through an intermediate idle wheel with the mirror-drum wheel *X*, which is sixteen times as large as *W*. In other words, each picture subtends an angle of $360/32 = 11\frac{1}{4}°$ and for a complete circuit of 360° the mirror-drum passes 32 mirrors and goes round half a revolution, and each mirror as it comes into use swings round or alters its angular inclination at half the angular rate at which each point of the film picture moves with respect to the first principal point of the lens.

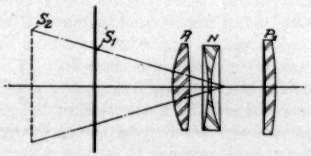


Fig. 13.

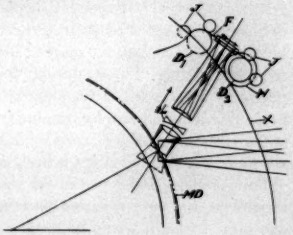


Fig. 14.

Variation in Picture Pitch

An important factor has now to be allowed for. This depends on the slight variations in picture pitch which are found to exist between different films. These, it is well known, cause imperfect and unsteady definition, even in the case of ordinary intermittent projection. Age and heat cause the film to shrink and we have found variations of picture pitch between ·7380 inch on the small side and ·7475 inch on the large side. If ·7430 inch is taken as a general mean value, these variations correspond to ± ·6 per cent., or 1/166 of the pitch.

Fig. 15 gives a section through the film drums D_1 and D_2 showing how, in the construction of D_2, there is incorporated a means whereby the picture on the screen can be raised or lowered at will without interrupting the running. This feature will be dealt with later. Both D_1 and D_2 are mounted on truly turned parallel axes or spindles finished to male cones at each end, forming adjustable bearings. *DP* is the driving pinion driven by the motor, and 9 is the pinion engaging directly or indirectly with the mirror-drum spur-wheel. Both *DP* and 9 are solidly fixed to the central spindle. 10 is a disc with toothed edge capable of running on an

independent bearing and carrying a loose pinion 11, concentrically placed, which gears into both spur-wheels 12 and 13, 12 being fixed to the central spindle and to 9, while 13 is capable of independent rotation on the spindle, but is coupled with the twin sprocket-wheels 5 and 6. Sprocket-wheel 5 is part of a long sleeve 5 and keyed to the latter is another sleeve and sprocket-wheel 6. Between sprocket-wheel 5 and sleeve 6 a weak coil spring tends to keep the sprocket-wheel 6 forced gently away from 5 with a force of 4 ounces or so. The object of this is to keep the film perforations always in end contact with the outer rounded ends of the teeth of sprocket 5 and thus prevent any capricious side play of the pictures.

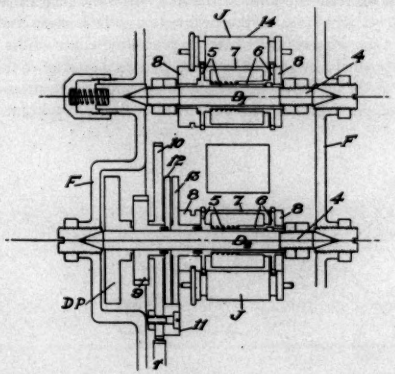


Fig. 15.

The circular rims of the sprocket-wheels 5 and 6 between and outside of the teeth are made ·03 to ·04 inch smaller in diameter than the loose rollers 7 and 8 on which the film rides while kept in contact by the pressure rollers 14 which are arranged to press the film gently home against the rollers 7 and 8 at two points 90° apart, just where the film passes on to the rollers and just where it begins to leave them. These jockey rollers are also shown in Fig. 16. The rollers 7 and 8 are turned to the exact circumference of four average picture pitches, so that the correct formula for their diameter is $\frac{4\,(\text{pitch})}{3{\cdot}14159} - {\cdot}006$, the last figure being the thickness of the film. When a correctly pitched film is being carried, these rollers simply revolve with the sprocket-wheels and central spindle so that there is no relative motion between the parts. If, however, the film pitch is, say, ·7380 inch, instead of the correct value of ·7430 inch, the free rollers together with the film slip back ·00125 inch as each pair of sprocket-teeth disengage themselves from the film. The loose rollers 7 and 8 on which the film rests enable this to be done without any of that dragging friction on the film which takes place with one-piece sprocket-drums.

Figs. 17 and 18 show, on an enlarged scale, how the short pitched film is only pulled forward, in the direction of the arrow, by the first pair of perforations, while the rear teeth do not touch the straight edges of the perforations. As soon as the teeth *a*, *a* disengage themselves, the film lags behind or slips back on the twin rollers until *b*, *b* take up the pull, and after *b*, *b* are disengaged then teeth *c*, *c* take up the pull, and so on. There are always four couples of perforations per picture pitch, and it is desirable that not more than four such couples shall mesh with the film at once, corresponding to a quarter of the drum's circumference. If, however, the pitch of the film picture is more than that of the rollers and sprocket, it is the rear teeth *d*, *d* which exercise the pull on the film while the forward teeth are more or less clear of the straight sides of the perforations. It is clear that as each pair of teeth, like *d*, *d*, come into contact with the preceding edges of the perforations, the film moves or is pushed forward on the free rollers by a quarter of the excess of pitch.

The result of this periodic retardation or advance by minute jerks taking place at the passage of every pair of sprocket-teeth is that every point in the projected

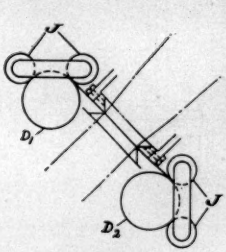


Fig. 16.

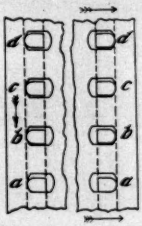


Fig. 17.

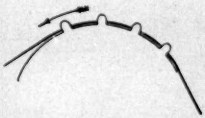


Fig. 18.

picture is vibrating up and down by a more or less visible amount. The angular amount of this vibration, if the film picture pitch is either ·005 inch long or short of the correct or average amount, will be 5·5 minutes for a projector lens of 3 inches focal length and 4·1 minutes for a lens of 4 inches focal length, the amount varying inversely as the focal length.

In the case of ordinary intermittent cinema projection, the accommodation of the film when out of pitch has to take place by sliding friction of the film on the pulling sprocket-drum, which is detrimental to the film as well as uncertain and capricious in its action. The best arrangement that can be devised for overcoming these deviations of the film pitch is shown in Figs. 19 and 20. Fig. 19 is a section through the lower sprocket-drum D_2 which pulls the film through the gate; 13 is the same differential spur-wheel as the 13 of Fig. 15, and all the mechanism to the left of 13 is exactly as in Fig. 15. In Fig. 19, however, the sprocket-drum is arranged to be expansible from a circumference of four pitches of ·736 inch up to a circumference of four picture pitches of ·750 inch, a variation in the circumference of $\cdot 014 \times 4 = \cdot 056$ inch. This corresponds to a

variation in radius of ·009 inch or about 2 per cent., thus allowing for 1 per cent. above the average pitch and 1 per cent. below. Riding stiffly upon the central spindle 4 is a truly-turned sleeve 16 in which are turned the two coned surfaces (whose slopes are at an angle whose tangent is 1/3) shown lying under the four pins about ·95 inch apart. These pins are fixed radially in the 16 radial sectors of the sprocket-drum 19 and project and fit through circular apertures drilled through the bobbin 18. The flanged ends of bobbin 18, shown in end view in Fig. 20, are cut with 16 equidistant radial slots forming guides into which project and fit the pins fixed into the ends of the radial sectors constituting the sprocket-drum 19. On the right of Fig. 20 one of the sprocket sectors is shown in section and shaded. The right-hand end of the sleeve 16 terminates in a divided head 16 A, engraved on its edge with marks for every ·1 per cent. variation in the diameter of the sprocket drum. The spindle 4 carries an enlargement or ring 17 threaded with 32 threads per inch just under the drum 16 A. This ring 17 is riveted to spindle 4 and carries a spring tongue catch 17 A with a rounded point pressing and engaging into holes

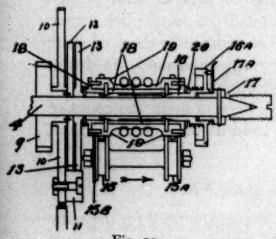


Fig. 19.

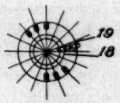


Fig. 20.

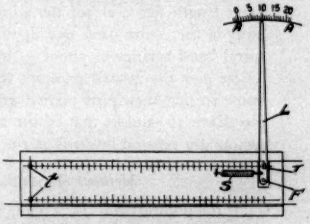


Fig. 21.

in the side of the divided drum 16 A, corresponding to the divisions and perhaps half divisions. This is sprung aside or outwards when the divided head 16 A is to be rotated and set to the picture pitch of the film that is to be run through the machine.

Gauge for measuring the Picture Pitch

Fig. 21 is a plan of the gauge for measuring the picture pitch. It consists of a shallow flat trough of 1·375 inches internal width and carrying two fixed teeth *t* at the left-hand end and a single tooth *T* fixed on a lever *L* at the other end. This lever rotates on the fulcrum *F*, while the point of its long arm sweeps round the graduated arc *AA*. When the lever is pointing to 10, the length from *t* to *T* should be 7·43 inches, which is 10 times the average pitch, there being 10 picture pitches of film or 40 perforations. The film should be held well down and stretched gently from left to right. A tension spring *S* always pulls the lever towards the left, and the tooth *T* is therefore pulled up against the left edge of the perforation through which it passes and registers any divergency of pitch on the arc *AA*. This error is enlarged 100 times if the lever ratio is 1 to 10 or 200 times if the ratio is 1 to 20.

Having in this simple way got the pitch of any film to be run through the projector, the divided head 16 A (Fig. 19) must be set to the same number as indicated on the arc *AA*. It is clear that, if the drum and sleeve 16 on the screw thread are rotated in the direction to force the sleeve towards the left, the two coned surfaces lift simultaneously all the 16 pairs of pins and cause all the 16 sectors of the drum and their pairs of teeth to radiate or spread symmetrically outwards and thus form a larger sized drum. Ordinarily the composite sprocket-drum 19 and the bobbin 18 which contains it, together with the coned sleeve 16 and the centre spindle 4, all rotate together as one piece. If, however, the differential gear (subsequently to be described) in Figs. 15 and 19 should be in operation, causing 13 to rotate upon the spindle 4, the sprocket-drum 19 and bobbin 18 rotate together with 13 upon the coned sleeve 16 as a parallel bearing as long as the differential gear is working. Thus the size of the sprocket-drum is not disturbed in the least. The coil spring 20 performs the necessary function of preventing any longitudinal movement of the bobbin 18, drum 19, and wheel 13 upon the parallel bearing formed by the coned sleeve 16, which, of course, has to have at least $\frac{1}{32}$ inch extra length left over for the adjustment. The sprocket-drum sectors are turned away in the centre parts to a depth of $\frac{1}{8}$ inch or so to accommodate three or more spiral band springs of about $\frac{1}{10}$ inch diameter lying ringwise around the drum. These give the inward pressure which is required for always keeping the sectors home so that their pins carried underneath are always bearing upon the cones of the sleeve 16, and so that, if the latter is withdrawn towards the right, the drum sectors are pressed inwards as the cones recede.

Methods of preventing Side Deviations of the Film

If the pulling drum D_2 is made radially expansible, there is no need for the same construction being applied to the following drum D_1, which can be constructed as in Fig. 15 where it is longitudinally expansible. A simpler construction, however, is now advisable for keeping the film pressed down on the drums D_1 and D_2 as well as for keeping the film always pressed gently towards the right so that the left-hand row of perforations is always brought in contact with the left-hand ends of the left-hand teeth on the drums D_1 and D_2.

I am quite aware that it is usual to expect a film to be controlled laterally in the gate by being confined edgeways between two straight edges, but since the breadth of films from edge to edge varies between 1·35 and 1·375 inches this method implies an amount of lateral looseness often fatal to steady projection. The method illustrated in Figs. 19, 22, and 23 is to be preferred. These show a jockey roller 15 and 15 A which by gentle spring pressure presses down the film on to the sprocket-wheels D_1 and D_2, the jockey rollers being all grooved to stride over the teeth with plenty of clearance. Fig. 22 shows that while D_1, which is passive, only requires one jockey roller 2, D_2 requires two jockey rollers 3 and 4, as the film on leaving D_2 is not under sufficient tension to keep home on the drum D_2, while on leaving D_1 the film is under tension owing to the pull of D_2. The jockey rollers 2 and 3 carry an extra projecting flange 15 B (Figs. 19 and 23) which laps over the extreme

edge of the film, and as the jockey rollers are gently pulled to the right (in Figs. 19 and 23) by a spring as well as pressed home against D_1 and D_2, these flanges like 15 B press the film on one side by contact with its edge until the left ends of the left perforations are kept in contact with the left-hand ends of the teeth of the sprocket-drums D_1 and D_2. The film is thus secured against any side deviations from true alignment and the pictures are kept true to the perforations. It must be remembered that in the case of a 4-inch projector lens a lateral deviation of only ·0012 inch (or 1/850) corresponds to 1 minute of arc of unsteadiness and 2 or 3 minutes are very perceptibly detrimental to good definition.

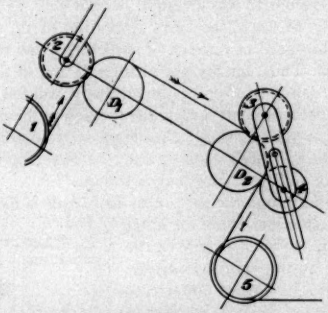


Fig. 22.

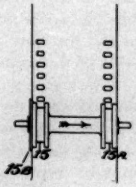


Fig. 23.

Fig. 24.

An alternative method of preventing side deviations of the film is illustrated in Fig. 24, which represents the right-hand end view of one of the sprocket-drums and the jockey-wheel pressing against it. This end of the sprocket-wheel has every alternate tooth missing, and in place of it a clearance hole. On the other hand, every alternate tooth of the jockey-wheel is missing and replaced by a clearance hole, so that the two can rotate together, each tooth in one being accommodated by the corresponding clearance hole in the other. Thus in the sprocket-drum there is a tooth to engage with every alternate perforation in the right-hand row of perforation, and the other alternate perforations are engaged by every alternate tooth on the jockey-wheel. Therefore, if the latter is pulled slightly to the right by spring pressure, it will by its teeth always exert a left to right pressure upon the film, and so cause the outer ends of the left-hand perforations to come into contact with the left-hand ends of the left-hand teeth of the sprocket-wheel.

Another variation of this method consists in replacing all the right-hand teeth of the sprocket-drum by clearance holes, and giving the jockey-wheel a complete right-hand set of teeth so that by means of its side pull it shall continually keep the film towards the right. In such case the pull of the film through the gate will be done entirely by the left-hand set of teeth on the sprocket-drum.

Method of raising or lowering the Screen Picture

Returning to Figs. 15 and 19 and the differential action for raising or lowering the screen picture, it will be seen that while the system of wheels and drums is operating, the pinion 11 forms a connecting link between spur-wheels 12 and 13 so that *DP*, 9, 10, 11, 12 and 13 all rotate as one piece. If the disc or wheel 10 is suddenly prevented from rotating by advancing a toothed detent against its edge, 12 goes on rotating as before, but causes the pinion 11 to rotate and drive round 13, while the latter carries round with it the film sprocket-drums to which it is attached. Obviously, if the number of teeth in 12 and 13 is equal, they will both be rotating together exactly as they were before 10 was arrested. If, however, 13 carries, say, 81 teeth while 12 carries only 80, then, for every revolution of 12, 13 will lose 1 tooth on it and therefore the speed of rotation of sprockets 5 and 6 will be decreased by 1/80 of the normal. Thus, for every 4 pictures ($\frac{1}{4}$ sec. of time usually) the picture on the screen will fall 1/20 of its height, so that it will fall one picture pitch in 5 secs. so long as the detent *r* is preventing the rotation of the disc 10. It is important that the bearing of 10 upon 12 shall be made moderately stiff and to that end a thin sprung washer can be placed concentrically between the flat faces of 10 and 12 in order to produce sufficient frictional resistance.

Fig. 16 is a vertical section through the gate. One half-blade is fixed true and square to the axis of the projector lens while the other half which fits it is pressed up against it evenly by the balanced lever shown in Figs. 25 and 26. Alternatively, the opposing gate surfaces may be curved concave to the lens to a radius of 4 × (E.F.L. of the lens) in order to neutralize the residual negative distortion of the projector lens, provided the latter is perfectly corrected against astigmatism, when the field is also curved to the radius of 4 × (E.F.L.). A pressure of about 4 ounces is brought to bear on the loose gate by means of the coil tension spring shown, which should be adjustable. Such a form of pressure gate is highly desirable in order to prevent capricious cockling of the film and to smooth out kinks and other defects, so that the film shall lie accurately in the focal plane or focal surface of the projector lens. The metal gate may even be replaced by a fixed polished plate of glass or quartz against which a pressure gate, also of glass or quartz, may be pressed, so that the film may be always held between the two inner surfaces of the glass or quartz plates. The metal gate may have the glass or quartz plates inserted into it.

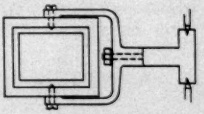


Fig. 25.

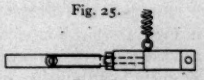


Fig. 26.

Mounting of Mirrors

We must now consider the question of the mounting of the mirrors around the mirror-drum, which is a most important matter, whose correct or incorrect carrying out may make or mar the whole scheme. Not only have the mirrors to be each adjustable to a sufficiently fine degree, but they must be quite free from strains and yet be firmly enough held in position to bear the pressure of ordinary

cleaning without perceptible disturbance. It is impossible to fix them after adjustment by using any solidifying plaster or cement, as its use inevitably leads to very serious mechanical strains and distortion of the surfaces. For the very best quality of cinema projection it is necessary that the mirrors shall be so flat and well figured that in each mirror no ray shall suffer any erroneous deviation relative to other rays greater than 1 minute of arc. The upper surface must be sufficiently parallel to the lower silvered surface, to ensure that the rays reflected from the upper unsilvered surface shall not deviate more than 1 to 3 minutes from the same rays reflected from the silvered surface, otherwise the faint image reflected from the upper surfaces will not properly coincide on the screen with the brighter image reflected by the silvered surface.

Fig. 27.

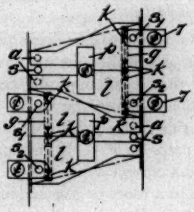


Fig. 29.

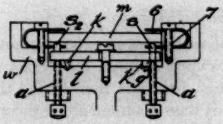


Fig. 28.

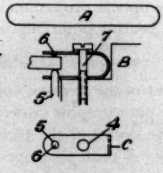


Fig. 30.

Fig. 27 is a plan and Fig. 28 a radial section of one of the mirrors mounted in a manner fulfilling the above conditions, when the mirrors are externally fitted. Both figures are on a reduced scale.

It is preferable to give to the mirrors a tapering form as shown in Figs. 27 and 29. Their mean diameter *dd* in the direction of movement should be about 1/8 of the E.F.L. of the projector lens to be used with them, the length not less than 5/16 of the E.F.L., the upper and lower margins enclosing an angle of 23°·5, and their thickness not less than 1/7 of their mean diameter. They can be of glass silvered behind, or of a reflecting metal.

The frame of the wheel is shown at *w* (Fig. 28) and *m* is the mirror. The bottom of the trough enclosing the mirrors may be turned cylindrical, but the V-shaped grooves *g*, *g* (Figs. 27 and 28) are each cut straight and parallel to each mirror, that is, at right angles to that radius of the drum which cuts the centre of each mirror.

Two independent rocking-frames or flat levers l, l are each provided underneath with two round-edged knife-edges or pins k, k which ride in the V-grooves, while on the upper side and near the outer corner, and beyond the knife-edges of each lever, are round-nosed studs s_1 and s_2, on each of which a corner of the mirror rests. The other and free ends of the levers rest upon the points of two fine-threaded adjusting screws a, a provided with lock nuts. These adjusting levers may be described as geared down, for the movement imparted to the mirror is a small fraction of the screw movement. At s, s are round-nosed studs which are fixed into the frame w; each stud forms the third support for each mirror. All these fixed studs s should be turned down to the same height in the lathe. At p, p are tongue springs which keep the two levers firmly pressed down into the V-bearings and against the adjusting screws.

Spring Clip

I have designed a form of spring clip (Fig. 30) carrying a pin or other pointed projection 6 to bear upon the outer surfaces of the mirrors opposite to the underneath bearing points 5 and so formed that it automatically centres itself upon 5 so that the true alignment or opposition between the upper and lower bearing points is more exactly secured. This spring clip is preferably made of clock spring steel band. The screw 7 regulates the pressure on 6. If any spring pressure point deviates by more than ·01 inch or so from correct opposition to the opposing point below the glass, strains on the mirrors to a more or less serious extent are apt to arise and mar the optical projection. A small spot of silvering should be cleared away where each bearing point impinges on the lower silvered mirror surface.

Adjustment of Mirrors

For adjustment of the mirrors, one method is to expose a moving picture film in a fixed cinema photographic camera upon a simple fixed black cross on a white ground and in the centre of the picture field and then thread this film into the projector machine. A small telescope is then mounted along the axis of projection AA (Fig. 3) and focussed upon the images of the black cross reflected by the mirrors as the film drum D_1 is very slowly rotated. The telescope should be provided with a diaphragm having a central circle 3 or 4 minutes in diameter and a high magnification of 50 or so. The image of the black cross, when any mirror is correctly adjusted, should keep within this circle as long as the mirror is in operation. A few minutes' operation of the adjusting screws a, a will speedily correct any errors of position.

Another method requires that the number of teeth in the film sprocket-drum pinion shall be an exact multiple of the four pictures which correspond to one rotation, such as 20, 24, 28, or 32 teeth. There may be, say, 8 teeth per mirror. Then a hinged detent or tongue may be so arranged that a mirror comes centrally opposite to the projector lens axis for every eighth tooth of the film sprocket-drum pinion, which can then be held in that position at every eighth tooth while the image of a black cross ruled in glass and fixed in the centre of the gate is reflected through the observing telescope. Each mirror in turn can be adjusted to reflect the image into the centre of the circle on the diaphragm of the telescope.

Relation between Number of Mirrors and B.F.L. of Projector Lens

As the exact relation between the number of mirrors mounted on the drum and the back focal length* (B.F.L.) of the projector lens is of vital importance, we here append Table III in which column 1 gives the B.F.L., column 2 the number of mirrors, column 3 a fraction expressing the horizontal length of the projected picture in terms of the length H of the hall from lens to screen, column 4 the minor oscillation, and column 5 the major oscillation. The B.F.L.'s are calculated to suit a picture pitch of exactly ·75 inch. Should the average picture pitch for which the machine is constructed be less than the standard ·75 inch, as, for instance, ·74 inch, the B.F.L.'s should be reduced by 1/75 of the value given in column 1. A minus correction has been made to the B.F.L.'s of the lenses in order to reduce the minor oscillation, as given in column 4, down to 38 per cent. This condition is also attained when the back focal length of the lens is made equal to

$$\frac{\frac{P}{2}}{\tan\left(\frac{360^\circ}{n}\right)},$$

in which P is the exact picture pitch and n is the number of mirrors on the mirror-drum.

Table III.

1	2	3	4	5
B.F.L. ins.	No. of mirrors	Length of picture / H	Minor oscillation − and + mins.	Major oscillation mins.
5·96	100	·168	·272	·55
5·242	88	·192	·405	·85
5·005	84	·200	·464	·92
4·764	80	·210	·533	1·07
4·524	76	·221	·623	1·25
4·286	72	·233	·733	1·48
4·046	68	·247	·874	1·75
3·807	64	·260	1·019	2·10
3·568	60	·277	1·23	2·5
3·328	56	·298	1·52	3·1
3·088	52	·321	1·92	3·9
2·968	50	·337	2·205	4·5
2·848	48	·351	2·50	5·2

Of course the equivalent focal length (E.F.L.) of the lenses can be got by deducting from the B.F.L.'s a correction equal to $(\text{B.F.L.})^2/H$ in which H is the length of the hall, in inches, measured from lens to screen. Fig. 31 illustrates how the above-mentioned formula for the back focal lengths (given in column 1 as an instance) works out. The vertical line *bc* represents the distance from the centre to the top of the gate, *ac* is the minor oscillation, and *cd* is the contrary oscillation due to the slight shortening of the focal length alluded to above, and implied in the

* The back focal length is the axial distance from the back principal point of the projector lens to the centre point of the film picture when the enlarged screen picture is in sharpest focus.

formula, which expresses the condition that any points in the picture arriving at the edge of the gate shall then suffer no displacement. The amounts of minor oscillation for any point in the picture as it travels up the gate are, of course, indicated by the horizontal distances from *bc* to the curve *ab*, while the amounts of oscillation due to the deduction from the theoretical back focal length, and implied in the formula, are indicated by the horizontal distances from *cb* to the straight line *bd*. Therefore the dotted curve is the resultant, and it will be found that the maximum distance or oscillation *nm* is about three-eighths of the initial minor oscillation *ac*, while a point at the edge of the gate suffers no displacement.

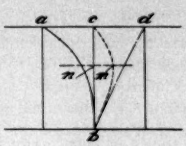


Fig. 31.

Mirror-drum with Mirrors fixed internally

So far I have described a mirror-drum so constructed that the mirrors are fixed externally and reflect outwards. I will now just allude to the other form of machine in which the mirrors of the drum are arranged to reflect inwards, but the designing of the necessary framework carrying the mechanism is rendered much more difficult. Fig. 4 is an elevation and Fig. 5 a sketch plan of such an arrangement. D_1 is an idle wheel introducing the film which twists through a right angle before passing round the toothed guide-wheel D_2, thence it passes through the gate to D_3, which is the driving drum that positively pulls the film along through the gate, then the film twists through a right angle again and passes out over the pulley D_4. L is the projector lens, whose axis is shown at an angle of about 30° with the perpendicular to any mirror m centrally opposed to it. Thus the axial reflection, as before, takes place in the plane of the mirror-drum A, and the central or axial ray is reflected outwards parallel to the mirror-drum's axis by means of the flat reflector R placed at an angle of 45° with the latter, or at a larger or smaller angle, if required. It is clear that the externally reflecting mirror-drum is the one to be adopted in practice.

Accuracy of Mechanism

The sprocket-wheels concerned in the apparatus must have their teeth cut to such accuracy that irregularities in motion in the mirror-drum leading to spasmodic displacements of the reflected rays greater than 1 minute of arc are avoided. None of these matters present difficulties that cannot be easily overcome.

Changes occurring between Exposures of Successive Pictures

Such a machine having been made and adjusted, it may be asked what would be the nature of the changes taking place in the screen picture when changing from one film picture to the next. Let the machine be moved round by hand very slowly while we watch the picture on the screen in relation to the position of the film pictures in the gate, and the mirrors in relation to the lens axis. Fig. 32 shows what happens.

Let a film picture fully and centrally occupy the gate, as in G_1. It is upside down, while at the same time the mirror lies centrally on the lens axis and reflects the major portion of the light to form the complete screen picture *ABCD*, but the neighbouring mirrors 1 and 3 are also receiving light and throw less bright but complete duplicate pictures on the screen above *AD* and below *BC*. The size of the screen, however, should be confined to the limits *ABCD*, preferably by a black frame; moreover, these supernumerary pictures can be almost completely cut off before reaching the screen by a rectangular aperture placed a few feet in front of the projector lens. Thus only the complete picture *ABCD* is allowed tc show itself.

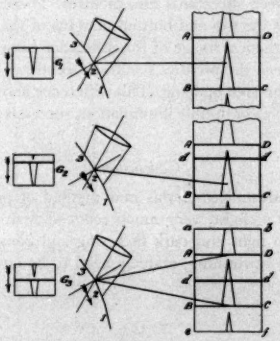


Fig. 32.

Let the machine be turned a little further so as to allow one-quarter of the next film picture to show in the top of the gate, as in G_2. Then mirror 1 will have about gone out of use, but mirror 3 is now receiving more of the light from the lens and throwing an image of increasing brightness up on to the top quarter of the screen above *dd*. Thus the three-quarter picture below *dd* is reflected by mirror 2 while the upper quarter picture is reflected by the oncoming mirror 3 and is an image of the upper quarter of the next film picture.

Turning the machine a little further, we then get the gate, as in G_3, occupied by half of one picture and half of the next, while mirrors 2 and 3 are equally in use. The result on the screen is that the line of demarcation *dd* between the two pictures has moved half-way down, and the complete screen picture is now built up of supplementary halves of two successive film pictures. The whole image of the film exposed in the gate and reflected by mirror 2 really should occupy on the screen the rectangular area *defd*, while the whole image of the film exposed in the gate and reflected by mirror 3 should occupy on the screen the rectangular area *addb*. Of these areas the halves *aADb* and *BefC* are superfluous and are of course

cut off and not allowed to show. Thus, on the screen, no sooner does the demarcation line *dd* reach the bottom than a new demarcation line begins to appear at the top and works downwards in its turn, and in this quiet and continuous way, the usual 16 pictures per second replace each other without there being any dark intervals or any of the clicking noise incidental to intermittent mechanisms.

The demarcation line *dd* may vary between an almost invisible line, if the vertical aperture of the gate is exactly equal to the pitch or distance (centre to centre) of the film pictures, and a sensibly broad dark line, if the gate aperture is sensibly narrower than the picture pitch. For it should be borne in mind that the demarcation line *dd* is not really an image of the intervals, either light or dark, which often exist between successive film pictures. These will be imaged on the screen as borders along the top and bottom margins of the screen picture, but *dd* is really a sort of hypothetical image of the difference between the correct picture pitch or distance between the pictures (centre to centre or feature to feature) and the vertical height of the gate opening. This difference shows on the screen either as dark or as narrow strips of double illumination, successively chasing one another down the screen.

Conclusion

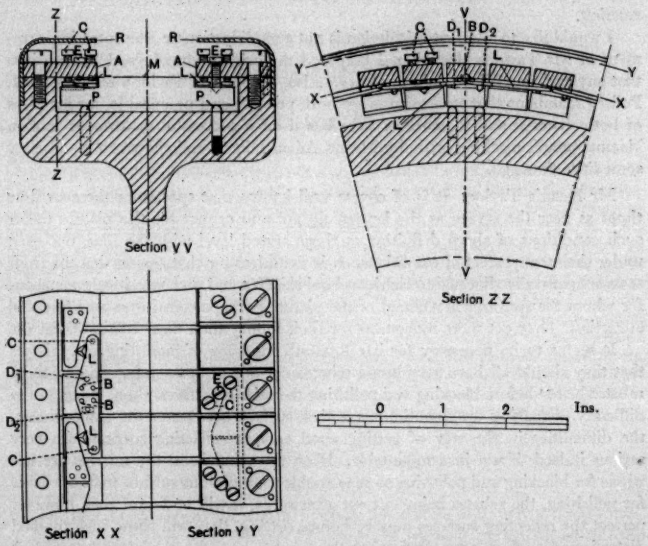
Finally, the great advantages of this new method of projection, in the shape of a substantial saving of light, very much reduced wear and tear of the films, complete freedom from light and dark flickering and consequent eye-strain, and much greater quietness in running, scarcely need to be emphasized.

DISCUSSION

Mr J. Guild: If I correctly understand the author, the calculations of the permissible amplitude of oscillations of the image points on the screen are all made in terms of the angle subtended by such displacements at the projecting lens. Is it not the distance from the screen to the nearest seats, rather than the distance to the lens, which should be considered in this connection? In most picture theatres, which are very long and narrow, the majority of the spectators occupy the front seats and are not distant from the screen by more than a quarter of the length of the hall.

Mr R. E. Reason: The method of mounting the mirrors, shown in the accompanying figure, was evolved in answer to an examination question set by Prof. Pollard, who suggested that the design might be of interest to Mr Taylor. Unfortunately, the dimensions given in the question were rather greater than those of the actual mirrors, and the design I showed at the meeting would, in consequence, have been difficult to manufacture. By a slight re-arrangement of the parts, however, I have found it possible to fulfil the necessary conditions without alteration in principle, and it is this design that I would like to advance now.

Three views of a portion of the drum are shown in the rough drawings. To the upper surface of the adjusting levers L are fixed two small steel balls B, B, which



engage with a trihedral hole and slot, stamped on the lower side of the arm A. About these the lever pivots, the required motion being imparted by the adjusting screw C, which bears on the opposite end of the lever. Studs D_1, D_2 are fixed alternately to the lower side of each lever and form supports for the mirror M. This is of steel; it has a slot cut at one end to engage with the longer stud D_2; D_1 rests on a plane, and a trihedral hole stamped at the opposite end engages with the fixed screw E. The mirror is kept in position by spring plungers P, one below each point of support. The pressure produced on the studs D_1, D_2 serves to keep the levers in place, for they are thus pinched between the mirror and supporting arms. Any effect of centrifugal force will tend merely to increase the stability of the system, and since both levers and mirrors are geometrically mounted, there will be no distortion due to unnecessary strain.

The general arrangement will be evident from the drawings. The arms A, of brass, are screwed and pinned to the cast iron drum. The adjusting levers are arranged in pairs at alternate ends of the mirrors, and are so shaped as to allow

the supports *E* of the intermediate mirrors to pass between them. *R*, *R* are light brass covers, fixed to the drum by small screws after the levers have been adjusted; they serve merely to protect the screws from accidental injury while the drum is rotating.

I would like to add that this design is not a modification or attempted improvement of Mr Taylor's, for when I prepared the original drawings, I had no idea that any such machine was in existence. Also, despite Mr Taylor's remarks, Prof. Pollard maintains that steel mirrors are both possible and practicable, for his ideas of both material and dimensions were based on a successful continuous motion kinematograph machine made by Messrs Adam Hilger, and shown to the Society some little time ago.

Mr Dennis Taylor: It is of course well known that spectators three or four times as near the screen as the lantern do not and cannot see the picture under such conditions of sharp definition as those seated level with the lens, not even under the most perfect of conditions. It is doubtless for that reason that the front seats are generally allocated to charity school children and such uncritical spectators, for whom the spurious movement of the picture perhaps constitutes an additional attraction. They get more movement for their money than those behind them do.

It seems to be necessary for Mr Reason's methods of mounting the mirrors that they should all have very broad rebates cut out of both ends. Should these rebates be cut before blocking and polishing the mirror surfaces, then, owing to the difficulty of making up a continuous optical surface because of the sunk rebates, the difficulties in the way of getting good enough reflecting surfaces are very serious indeed, if not insurmountable. If on the other hand the mirrors are left whole for blocking and polishing so as to enable a continuous surface to be available for polishing, the rebates being cut out afterwards, it will be found that, however perfect the reflecting surfaces may be before cutting, they will show such marked distortion of surface, after cutting out the rebates, as to be in most cases useless; these distortions, due to relief of mechanical tension in the material, take place whether the mirrors are made of glass or of chromium steel. These difficulties are familiar to all opticians experienced in the working of fairly high-class plane reflecting mirrors.

Another gentleman said he considered that the adjustable mounting and supports for the mirrors which I described would be quite impossible to carry out within the limits of size that I had specified for them in the case of the smaller focal lengths of projector lenses, such as 3 inches. I should point out, however, that the lengths of the levers employed lie across the mirror-drum or parallel to the axis of rotation of the latter and therefore their length can be made as much longer than the greater length of the mirrors as is found convenient. For a 3-inch projecting lens the mirrors will be at least ·5 inch wide in the plane of rotation and 1 inch or so parallel to the axis of rotation, while the levers can be 2 inches long if desired.

THE ADDITION OF ABERRATIONS

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ABSTRACT. The evaluation of the aberration coefficients of a complex system in terms of those of its components is investigated on a general basis which is independent of special properties of the system. For example, in an axially symmetrical system the refracting surfaces may be any surfaces of revolution, or the media may be non-homogeneous as long as they are homogeneous for all circles normal to and centered on the axis. Two methods are employed. The first, by successive approximation applied to equations of reduced order, is utilized to obtain the well-known results for the six first order aberrations and the comparatively unknown results for the ten second order aberrations. For higher orders this method becomes almost impracticable. The second method is based upon the evaluation of a function when it is stationary by Lagrange's theorem, and gives a theoretical solution to the problem for all orders. The results already obtained by the previous method are checked by this more general method, and the addition formulae for the fifteen third order aberrations are also obtained. These are given in a simplified form free from all removable terms by the definition of a new function in terms of which the eikonal is expressed. The conditions for freedom from aberrations are given in terms of this function.

Introduction

The theory of optical instruments, particularly in the parts dealing with aberrations, is complicated by the great number of degrees of freedom present in all but the very simplest systems of lenses. This complexity is one of the factors most effective in retarding the transformation of lens design from an empirical footing to a state in which the evolution of a system is based on a reasoned scientific procedure. If such a transformation is to take place, two conditions must be satisfied—the designer must have a thorough understanding of the properties of simple classes of lenses, and in addition a knowledge of the way in which the properties of the complete instrument depend on those of its elements. Since he normally makes extensive use of components the properties of which he has learnt from previous experience, the first condition is often satisfied; but even in these cases it appears that the knowledge already gained is not fully utilized, as he finds it necessary to trace a series of rays through the entire system before he is confident that its performance will fulfil his expectations. In other words a knowledge of the properties of the components does not enable him to derive those of the complete instrument with the necessary precision. The present paper describes an attempt to bridge the gap which this customary procedure reveals.

An initial limitation to the problem may be made by assuming that the systems

to be considered are symmetrical about an axis*. This restriction is almost invariably adopted by designers in their calculations, and, although it does not represent exactly the true state of affairs in actual instruments, the assumption is reasonable since the departures from axial symmetry in well constructed examples are small enough for their effects on the performance of the system to be negligible. A further simplification is made by considering in the first place a system which is divided into two parts only, the general expressions following from repeated application of the formulae for this special case. It is convenient also to assume that both parts of the system are surrounded by a medium of unit refractive index. This assumption will not hold for some of the applications of the formulae to be proposed hereafter, but the necessary generalization can be made immediately when the final results for the simplified case have been found, and the course proposed leads to a distinct economy in symbols. As usual in aberrational investigations of a general character, a potential function is employed. In this enquiry the eikonal will be adopted; the principles followed, however, are applicable to any of the functions available. Use will be made of a number of results contained in a recent paper†. For typographical simplicity some departures from the notation of that paper will be made.

The investigation will be followed more readily if the general arrangement of these notes is indicated here. Two methods are utilized, both based on the same fundamental relations. The first is the direct method of successive approximation, and is applied to equations of lower order than the fundamental relations to obtain the addition formulae of the first and second orders. Its use is simplified by so choosing the variables of the components that they are in all cases equal to a first approximation to the corresponding variables for the combined system. This method becomes troublesome in dealing with aberrations of high orders, and to avoid the difficulties attendant on it, the second method, which yields the terms of a general order, is introduced. The general formula is most simply expressed in terms of the original variables, but the addition formulae are both more concise and more useful in terms of the variables employed in the investigation by the first method, and it is in this form that they are given. A modification of the form assumed as the standard expression for the eikonal enables the addition formulae to be simplified by the omission of terms involving the products of the aberrational coefficients of lower orders.

The fundamental equations

Rectangular axes of co-ordinates are taken in the object space, the image space, and the intermediate space which separates the two components into which the system is assumed to be divided. In each case the x axis coincides with the axis of symmetry of the system of refracting surfaces, and the various y axes lie in a common plane through this axis, so that the several z axes are in another axial

* In a few special instruments large departures are deliberately made from axial symmetry. The principles followed in this discussion are applicable and sufficient for the construction of a complete theory for such instruments, but this general case is not of sufficient importance to justify its adoption for very detailed discussion.

† "The Changes in Aberrations when the Object and Stop are moved." *Trans. Opt. Soc.* 23 (1921–22) 311. References to this paper will be made in the abbreviated form *O.S.* 23.

plane. The direction cosines of a typical ray in the space preceding the entire system will be denoted by L, M, N, in the intermediate space by L', M', N', and in the final image space by L'', M'', N''. Let a point be taken on the axis in each of these three spaces—say X, X', X'' respectively. If these are the reference points for which the various potential functions are constructed, the value of the eikonal for the first portion of the system represents the optical length of the ray path between the feet of the perpendiculars to the ray from X and X'; the value for the second component represents the length along the ray between the feet of the perpendiculars from X' and X''; and that for the whole length between the perpendiculars from X and X''. The value of the eikonal for the whole is thus the sum of the values for the parts.

Denote the three functions as follows:

for the first part by ϵ;
for the second part by e;
for the whole by $\mathscr{E}$.

Then ϵ is a function of $M^2 + N^2$, $MM' + NN'$, $M'^2 + N'^2$;
e is a function of $M'^2 + N'^2$, $M'M'' + N'N''$, $M''^2 + N''^2$;
and $\mathscr{E}$ is a function of $M^2 + N^2$, $MM'' + NN''$, $M''^2 + N''^2$.

Their values satisfy the relation

$$\epsilon + e - \mathscr{E} = 0 \qquad \ldots\ldots(1),$$

and this equation becomes an identity if each function is expressed in terms of the same variables. The relations which exist between the six direction cosines M, N; M', N'; M'', N'' express the conditions that the ray defined by any two pairs is the same as is defined by any other two pairs. They are found from the fundamental property of the potential functions that the unknown ray co-ordinates may be obtained by differentiating the function with respect to the independent variables. Thus if the ray meets the planes normal to the axis through X, X', X'' in points whose co-ordinates are (y, z), (y', z'), and (y'', z'') respectively, it follows from the properties of the eikonal that

$$y = \frac{\partial \epsilon}{\partial M}, \quad z = \frac{\partial \epsilon}{\partial N}, \quad y' = -\frac{\partial \epsilon}{\partial M'}, \quad z' = -\frac{\partial \epsilon}{\partial N'};$$

$$y' = \frac{\partial e}{\partial M'}, \quad z' = \frac{\partial e}{\partial N'}, \quad y'' = -\frac{\partial e}{\partial M''}, \quad z'' = -\frac{\partial e}{\partial N''};$$

$$y = \frac{\partial \mathscr{E}}{\partial M}, \quad z = \frac{\partial \mathscr{E}}{\partial N}, \quad y'' = -\frac{\partial \mathscr{E}}{\partial M''}, \quad z'' = -\frac{\partial \mathscr{E}}{\partial N''}.$$

The elimination of y, z, y', z', y'', z'' gives relations which may be written in the brief form

$$\frac{\partial (\epsilon + e - \mathscr{E})}{\partial m} = 0 \qquad \ldots\ldots(2),$$

where m represents any one of the six direction cosines used in the construction of the potential functions. Any corresponding pair of these relations suffices to

yield a theoretically complete solution of the present problem*, and the final result is necessarily independent of the particular choice made. As the immediate aim of this enquiry is to express the coefficients of $\mathscr{E}$ in terms of the coefficients of ϵ and e, the most direct course is to evaluate M' and N' in terms of M, N, M'', and N'' and to eliminate them from ϵ and e, utilizing as the basis the special cases when m in equation (2) is replaced in turn by M' and N'.

Before proceeding with these operations it is convenient to take new variables in harmony with the accepted definition of aberrations†. Let these be ξ, ψ, θ in ϵ; k, p, t in e; K, P, T in $\mathscr{E}$, the values being given by regarding the following equations as identities in n‡:

$$\left.\begin{aligned}(\sigma-\gamma)^2(\xi,\psi,\theta\,[\!]\,1,n)^2 &= \{M-\gamma M'+n(M-\sigma M')\}^2+\{N-\gamma N'+n(N-\sigma N')\}^2\\ (s-g)^2(k,p,t\,[\!]\,1,n)^2 &= \{M'-gM''+n(M'-sM'')\}^2+\{N'-gN''+n(N'-sN'')\}^2\\ (S-G)^2(K,P,T\,[\!]\,1,n)^2 &= \{M-GM''+n(M-SM'')\}^2+\{N-GN''+n(N-SN'')\}^2\end{aligned}\right\} \quad \ldots.(3),$$

where for the moment $\sigma, \gamma, s, g, S, G$ may be regarded as independent quantities. The application of (2) in the chosen cases gives

$$\begin{aligned}&\{(M-\gamma M')(2\gamma\epsilon_\xi+\sigma\epsilon_\psi)+(M-\sigma M')(\gamma\epsilon_\psi+2\sigma\epsilon_\theta)\}(\sigma-\gamma)^{-2}\\ &\quad-\{(M'-gM'')(2e_k+e_p)+(M-sM'')(e_p+2e_t)\}(s-g)^{-2}=0 \qquad \ldots.(4),\end{aligned}$$

with a similar equation with N's in place of M's, the suffixes of course indicating differentiation with respect to the subscript variables. Now let S and G be expressible in terms of the other quantities by the relations

$$\sigma s = S, \quad \gamma g = G \qquad \ldots.(5).$$

Then (4) may be arranged in the form

$$(S-G)M'-(s-g)M-(\sigma-\gamma)sgM''=\frac{(M-GM'')A+(M-SM'')B}{2C} \qquad \ldots.(6),$$

where

$$\left.\begin{aligned}A &= \frac{2\gamma\epsilon_\xi+\sigma\epsilon_\psi}{\sigma-\gamma}s-\frac{2e_k+e_p}{s-g}\\ B &= \frac{\gamma\epsilon_\psi+2\sigma\epsilon_\theta}{\sigma-\gamma}g-\frac{e_p+2e_t}{s-g}\\ C &= \frac{\gamma^2\epsilon_\xi+\gamma\sigma\epsilon_\psi+\sigma^2\epsilon_\theta}{(\sigma-\gamma)^2}-\frac{e_k+e_p+e_t}{(s-g)^2}\end{aligned}\right\} \qquad \ldots.(7).$$

Now

$$\begin{aligned}&(S-G)M'-(s-g)M-(\sigma-\gamma)sgM''\\ &\quad=-\{(S-G)(M-\gamma M')-(\sigma-\gamma)(M-GM'')\}\gamma^{-1}\\ &\quad=-\{(S-G)(M-\sigma M')-(\sigma-\gamma)(M-SM'')\}\sigma^{-1}\\ &\quad=(S-G)(M'-gM'')-(s-g)(M-GM'')\\ &\quad=(S-G)(M'-sM'')-(s-g)(M-SM''),\end{aligned}$$

* This was noted by both Hamilton and Bruns, but neither applied the method in a useful form.

† *O.S.* 23, p. 314.

‡ For explanation of the notation see *O.S.* 23, p. 317.

and therefore by (3) and (6)

$$\left.\begin{aligned}(\xi, \psi, \theta)(1, n)^2 &= (K, P, T)\left(s - \frac{(\gamma + \sigma n)\,A}{2(\sigma - \gamma)\,C},\quad gn - \frac{(\gamma + \sigma n)\,B}{2(\sigma - \gamma)\,C}\right)^2\\ (k, p, t)(1, n)^2 &= (K, P, T)\left(1 + \frac{(1 + n)\,A}{2(s - g)\,C},\quad n + \frac{(1 + n)\,B}{2(s - g)\,C}\right)^2\end{aligned}\right\} \ldots(8).$$

These equations may be used to express $\xi, \psi, \theta, k, p, t$ in terms of K, P, T by successive approximation. It will be noted that they satisfy the relations

$$\begin{aligned}\sigma^2\xi - 2\sigma\gamma\psi + \gamma^2\theta &= S^2K - 2SGP + G^2T,\\ k - 2p + t &= K - 2P + T,\\ \xi - 2\psi + \theta &= s^2k - 2sgp + g^2t,\end{aligned}$$

which in the original notation are identities.

Equations of lower order

When $\xi, \psi, \theta, k, p, t$ have been obtained, instead of substituting in the original equation (1) it is advantageous in dealing with aberrations of low orders by the direct method to make use of equations of a lower order. These may be derived by noting that

$$\Sigma m \frac{\partial}{\partial m}(\epsilon + e - \mathscr{E}) = 0 \qquad \ldots.(9),$$

where the summation sign indicates that m is to receive in turn each of the six values for which equation (2) has been shown to be true. Since $\xi, \psi, \theta, k, p, t$, K, P, T are homogeneous functions of the same order in these variables, equation (9) may be written

$$\xi\epsilon_\xi + \psi\epsilon_\psi + \theta\epsilon_\theta + ke_k + pe_p + te_t - K\mathscr{E}_K - P\mathscr{E}_P - T\mathscr{E}_T = 0 \ldots.(10).$$

In this equation substitute for $\xi, \psi, \theta, k, p, t$ from (8). The resulting relation equates to zero the sum of the products of certain coefficients with K, P, and T respectively. As these are independent variables, this must be taken to imply that the value of each of the three coefficients is zero. Hence the relations desired, viz.:

$$\left.\begin{aligned}\mathscr{E}_K &= s^2\epsilon_\xi + e_k - \frac{A^2}{4C}\\ \mathscr{E}_P &= sg\epsilon_\psi + e_p - \frac{AB}{2C}\\ \mathscr{E}_T &= g^2\epsilon_\theta + e_t - \frac{B^2}{4C}\end{aligned}\right\} \qquad \ldots.(11).$$

All the quantities appearing in these equations involve only the first differentials of those in equation (1). It follows that to obtain the desired formulae for the addition of aberrations it is sufficient to find approximations for $\xi, \psi, \theta, k, p, t$ of one order lower than equation (1) appears to indicate. It may be noted that the substitution of equation (11) for (1) corresponds to the substitution of ray positions for differences of optical path, and the advantages and disadvantages of the change are analogous in the two cases.

Paraxial relations

So far γ, σ, g, and s have been regarded as arbitrary independent quantities. In the relations of actual interest they are connected in definite ways which admit of a simple physical interpretation. Previous results indicate that they represent transverse magnifications*. One pair, say γ and g, may be associated with the points X, X', X'', for which the partial and complete eikonals have been defined. Instead of a casual selection suppose that these three points bear to one another an object and image relation for paraxial rays of the wave-length under consideration. That is to say, if X is the point in which a small transverse object meets the axis in the object space, let the linear dimensions of its image (situated at X') in the first portion of the system be γ times as great, and similarly those of the image in the whole instrument, formed at X'', be G times those of the original object. Since $G = \gamma g$, it follows that g is the magnification for paraxial rays in the second portion of the system for an object at X' whose image is at X''. First order terms in ϵ, e, and $\mathscr{E}$ involving ψ, θ, p, t, P, T are then absent, and apart from constant terms which have no physical importance,

$$\left.\begin{aligned} \epsilon &\underset{1}{\approx} \tfrac{1}{2}\xi\phi\,(\sigma-\gamma)^2\gamma^{-1} \\ e &\underset{1}{\approx} \tfrac{1}{2}kf\,(s-g)^2g^{-1} \\ \mathscr{E} &\underset{1}{\approx} \tfrac{1}{2}KF\,(S-G)^2G^{-1} \end{aligned}\right\} \quad \ldots.(12)\dagger,$$

where ϕ, f, and F are the focal lengths of the two components and of the complete lens respectively. It follows from (7) that B vanishes to a first approximation, and also that to this order both sides of the second and third of equations (11) are zero. It is convenient to secure a zero value for A also to this approximation. By (7) and (12) this involves the condition

$$\phi\,(\sigma-\gamma) = f\,(-s^{-1}+g^{-1}) \quad \ldots.(13),$$

which implies that σ, s, and S are magnifications in the first component, second component, and the complete instrument for a given object in the object space. They may be regarded as the paraxial transverse magnifications of stop positions. All the arbitrary quantities that have been introduced now bear definite meanings.

It is important to note that C remains finite when A and B vanish to the first order, (7) giving

$$2C \underset{1}{\approx} \phi\gamma + fg^{-1} \quad \ldots.(14).$$

Also from the first of equations (11)

$$\begin{aligned} F\,(S-G) &= \{\phi\,(\sigma-\gamma)^2s^2\gamma^{-1} + f\,(s-g)^2g^{-1}\}\,G\,(S-G)^{-1} \\ &= \phi\,(\sigma-\gamma)\,sg = f\,(s-g) \end{aligned} \quad \ldots.(15),$$

by (5) and (13). These are well-known results in the elementary theory of optical instruments, and are usually found from a collinear theory of imagery. They are noticed here to illustrate that equations (8) and (11) involve all the relations which subsist between the parts and the entire instrument.

* *O.S.* 23, pp. 313–4.

† *O.S.* 23, equation (12). $\underset{n}{\approx}$ is used to denote "equals to the nth approximation."

First order aberrations

Since to the first order A and B vanish while C is finite, A and B are negligible in finding the addition laws for first order aberrations. The appropriate substitutions are by (8)

$$\left.\begin{array}{lll} \xi \underset{1}{=} Ks^2, & \psi \underset{1}{=} Psg, & \theta \underset{1}{=} Tg^2 \\ k \underset{1}{=} K, & p \underset{1}{=} P, & t \underset{1}{=} T \end{array}\right\} \quad \ldots.(16).$$

Inserting these values in (11) and using the triple suffix notation identifying the aberrations by the order, series, and term number*, but retaining the distinctive characters ϵ, e, and $\mathscr{E}$, the results are, for series 0,

$$\mathscr{E}_{20i} = \epsilon_{20i} s^{4-i} g^i + e_{20i} \quad (i = 0, 1, 2, 3, 4) \qquad \ldots.(17),$$

and for the aberration of series 1,

$$\mathscr{E}_{212} (S - G)^2 = \epsilon_{212} (\sigma - \gamma)^2 s^2 g^2 + e_{212} (s - g)^2,$$

which by (15) may be written

$$\mathscr{E}_{212} F^{-2} = \epsilon_{212} \phi^{-2} + e_{212} f^{-2} \qquad \ldots.(18).$$

The addition laws for these aberrations have been previously expressed in the form (17), and (18) will be recognized as the usual formula for the Petzval sum since $\mathscr{E}$, ϵ, and e contain F, ϕ, and f respectively as their dimensional factors.

Modified notation

Before carrying the approximations to a higher order it is convenient to make a small change in the notation for the leading portion of the system. Take, in place of ξ, ψ, θ, new variables κ, π, τ defined by

$$\xi = \kappa s^2, \quad \psi = \pi sg, \quad \theta = \tau g^2 \qquad \ldots.(19),$$

and let η be used to denote ϵ when this change has taken place. The principal equations may now be written

$$\eta + e - \mathscr{E} = 0 \qquad \ldots.(20),$$

$$\left.\begin{array}{l} A = (2G\eta_\kappa + S\eta_\pi)\,\phi - (2e_k + e_p) f \\ B = (G\eta_\pi + 2S\eta_\tau)\,\phi - (e_p + 2e_t) f \\ C = (G^2\eta_\kappa + SG\eta_\pi + S^2\eta_\tau)\,\phi^2 + (e_k + e_p + e_t) f^2 \end{array}\right\} \qquad \ldots.(21),$$

$$\left.\begin{array}{l} (\kappa, \pi, \tau \overset{1}{\wedge} 1, \eta)^2 = (K, P, T)\left\{1 - \dfrac{(G + Sn)\,A\phi}{2C},\ n - \dfrac{(G + Sn)\,B\phi}{2C}\right)^2 \\ (k, p, t \overset{1}{\wedge} 1, \eta)^2 = (K, P, T)\left\{1 + \dfrac{(1 + n)\,Af}{2C},\ n + \dfrac{(1 + n)\,Bf}{2C}\right)^2 \end{array}\right\} \qquad (22),$$

$$\left.\begin{array}{l} \mathscr{E}_K = \eta_\kappa + e_k - \dfrac{A^2}{4C} \\ \mathscr{E}_P = \eta_\pi + e_p - \dfrac{AB}{2C} \\ \mathscr{E}_T = \eta_\tau + e_t - \dfrac{B^2}{4C} \end{array}\right\} \qquad \ldots.(23),$$

* *O.S.* 23, p. 320.

where it should be noted that the new A, B, and C are not identical with the old. The terms in η of order n and series m are

$$c_n \{(\kappa\tau - \pi^2)(S-G)^2F^2\phi^{-2}\}^m (\eta_{n,m,2m}, \eta_{n,m,2m+1}, \cdots \eta_{n,m,2n-2m} \mathfrak{Q} \kappa, -2\pi, \tau)^{n-2m} \quad \ldots.(24),$$

where
$$c_n = -\frac{(2n-2)!}{2^{2n-1}(n-1)!\,n!},$$

and
$$\eta_{n,m,i} = \epsilon_{n,m,i} s^{2n-2m-i} g^{i-2m} \quad \ldots.(25).$$

The terms in e corresponding to (24) are obtained by replacing η, κ, π, τ, ϕ by e, k, p, t, f respectively; in $\mathscr{E}$ they are replaced by $\mathscr{E}$, K, P, T, F respectively. The first order addition formulae (17) and (18) in this notation become

$$\mathscr{E}_{20\,i} = \eta_{20\,i} + e_{20\,i} \quad \ldots.(26),$$
$$\mathscr{E}_{212}F^{-2} = \eta_{212}\phi^{-2} + e_{212}f^{-2} \quad \ldots.(27).$$

Second order aberrations

The aberrations of higher orders may be obtained by the direct process of carrying the various approximations one step forward at a time, making use of equations (21), (22), and (23). The task, however, may be greatly lightened by the use of sufficiently obvious devices which either lead to the removal of some terms or enable numerical relations to be established between others so that all have not to be evaluated independently. For example, instead of using equation (23) to find the laws of combination for second order aberrations, it is simpler to use (20) and

$$\kappa\eta_\kappa + \pi\eta_\pi + \tau\eta_\tau + ke_k + pe_p + te_t - K\mathscr{E}_K - P\mathscr{E}_P - T\mathscr{E}_T = 0 \ldots.(28),$$

which corresponds in the revised notation to (10). Denote by $(\mathscr{E})_n$ the terms in $\mathscr{E}$ of the nth order in K, P, T, and similarly for η and e when expressed in terms of their own natural variables. Then neglecting the constant term,

$$\mathscr{E} = (\mathscr{E})_1 + (\mathscr{E})_2 + (\mathscr{E})_3 + \ldots,$$

and by the properties of homogeneous functions

$$K\mathscr{E}_K + P\mathscr{E}_P + T\mathscr{E}_T = (\mathscr{E})_1 + 2(\mathscr{E})_2 + 3(\mathscr{E})_3 + \ldots,$$

so that by (20) and (28),

$$\begin{aligned} -(\mathscr{E})_1 + (\mathscr{E})_3 + 2(\mathscr{E})_4 + \ldots &= K\mathscr{E}_K + P\mathscr{E}_P + T\mathscr{E}_T - 2\mathscr{E} \\ &= -(\eta)_1 + (\eta)_3 + 2(\eta)_4 + \ldots \\ &\quad -(e)_1 + (e)_3 + 2(e)_4 + \ldots. \end{aligned}$$

Now by (13), (19), and (22),

$$(\eta)_1 + (e)_1 = (\mathscr{E})_1 + \frac{KA^2 + 2PAB + TB^2}{8C^2} F^2 (S-G)^2 (\phi\gamma + fg^{-1}),$$

and by (21),
$$C \equiv \tfrac{1}{4} F^2 (S-G)^2 (\phi\gamma + fg^{-1}),$$

so that
$$(\mathscr{E})_3 = (\eta)_3 + (e)_3 - \frac{KA^2 + 2PAB + TB^2}{4C},$$

where each variable receives its first approximation value and only the lowest order terms are retained in A, B, C. The values to be substituted are therefore

$$A \fallingdotseq -\tfrac{1}{2}F(S-G)\{(a_0, a_1, a_2 \text{\textbar\textbar} K, -2P, T) + \tfrac{1}{2}(S-G)(a_1', a_2' \text{\textbar\textbar} P, T)\},$$
$$B \fallingdotseq \tfrac{1}{2}F(S-G)\{(a_1, a_2, a_3 \text{\textbar\textbar} K, -2P, T) + \tfrac{1}{2}(S-G)(a_1', a_2' \text{\textbar\textbar} K, P)\},$$
$$C \fallingdotseq \tfrac{1}{2}\phi f F(S-G)^2,$$

where
$$\left.\begin{aligned} a_i F(S-G) &= (\eta_{20i} G - \eta_{20(i+1)} S)\,\phi - (e_{20i} - e_{20(i+1)})\,f \\ a_1' F^{-1} &= -\eta_{212} S\phi^{-1} + e_{212} f^{-1} \\ a_2' F^{-1} &= \eta_{212} G\phi^{-1} - e_{212} f^{-1} \end{aligned}\right\} \quad \ldots.(29).$$

The desired relations are therefore, for series 0,

$$\left.\begin{aligned} \mathscr{E}_{300} &= \eta_{300} + e_{300} + a_0{}^2.2F/\phi f \\ \mathscr{E}_{301} &= \eta_{301} + e_{301} + a_0 a_1.2F/\phi f \\ \mathscr{E}_{302} &= \eta_{302} + e_{302} + (3a_1{}^2 + 2a_0 a_2).2F/5\phi f \\ \mathscr{E}_{303} &= \eta_{303} + e_{303} + (9a_1 a_2 + a_0 a_3).F/5\phi f \\ \mathscr{E}_{304} &= \eta_{304} + e_{304} + (3a_2{}^2 + 2a_1 a_3).2F/5\phi f \\ \mathscr{E}_{305} &= \eta_{305} + e_{305} + a_2 a_3.2F/\phi f \\ \mathscr{E}_{306} &= \eta_{306} + e_{306} + a_3{}^2.2F/\phi f \end{aligned}\right\} \quad \ldots.(30),$$

and for series 1,

$$\left.\begin{aligned} \frac{\mathscr{E}_{312}}{F^2} &= \frac{\eta_{312}}{\phi^2} + \frac{e_{312}}{f^2} + \left\{\frac{4(a_0 a_2 - a_1{}^2)}{5(S-G)^2} + \frac{a_0 a_2' + a_1 a_1'}{S-G} + \frac{a_1'^2}{4}\right\}\frac{2}{\phi f F} \\ \frac{\mathscr{E}_{313}}{F^2} &= \frac{\eta_{313}}{\phi^2} + \frac{e_{313}}{f^2} + \left\{\frac{2(a_0 a_3 - a_1 a_2)}{5(S-G)^2} + \frac{a_1 a_2' + a_2 a_1'}{S-G} - \frac{a_1' a_2'}{4}\right\}\frac{2}{\phi f F} \\ \frac{\mathscr{E}_{314}}{F^2} &= \frac{\eta_{314}}{\phi^2} + \frac{e_{314}}{f^2} + \left\{\frac{4(a_1 a_3 - a_2{}^2)}{5(S-G)^2} + \frac{a_2 a_2' + a_3 a_1'}{S-G} + \frac{a_2'^2}{4}\right\}\frac{2}{\phi f F} \end{aligned}\right\} \ldots(31).$$

It will be shown later that these second order relations can be expressed in a form which greatly simplifies the calculations.

Further approximations

The methods so far illustrated, though quite suitable in dealing with aberrations of low orders, are inconvenient when expressions are required for the aberrations of high orders. To deal with such cases it is preferable to use a transformation of (22) obtained by changing the value of n,

$$\left.\begin{aligned} (K, P, T \text{\textbar\textbar} 1, n)^2 &= (\kappa, \pi, \tau \text{\textbar\textbar} 1 + \frac{(G+Sn)\,AF}{if^2},\quad n + \frac{(G+Sn)\,BF}{if^2})^2 \\ &= (k, p, t \text{\textbar\textbar} 1 - \frac{(1+n)\,AF}{j\phi^2},\quad n - \frac{(1+n)\,BF}{j\phi^2})^2 \end{aligned}\right\} \ldots(32),$$

where
$$\left.\begin{aligned} i &= (2G\eta_\kappa + S\eta_\pi)\,g + (G\eta_\pi + 2S\eta_\tau)\,s \\ j &= (2e_k + e_p)\,g + (e_p + 2e_t)\,s \end{aligned}\right\} \quad \ldots.(33).$$

Each of equations (32) is of a form similar to

$$\kappa = K + h,$$

where h is a function of known form into which only the six variables $\kappa, \pi, \tau, k, p, t$ enter explicitly. From equation (20) they therefore enable $\mathscr{E}$ to be expressed in terms of K, P, T by the direct application of Lagrange's expansion theorem for

several variables. This method has the advantage of avoiding the troublesome process of finding successive approximations to the values of the initial variables. In dealing with as many as six separate variables care has to be taken in writing down the terms of the expansion to see that none are omitted—a mistake very easily made with this particular formula. For this and other reasons it is better to return to the fundamental equations as described below.

The basis of a more general investigation

The preceding development of the aberration relations is the one which yields the simplest expressions for the first order aberrations, but is open to the objection that the component expressions involve the positions of both object and stop. There are advantages in adopting a system in which these only appear in coefficients in a combination of quantities which are free from such co-ordinates. The investigation of addition formulae may be made in a very general way by retaining the direction cosines to be eliminated as variables. This course offers the additional advantage that there are only two variables instead of six to deal with, and is particularly convenient when Lagrange's theorem is to be employed to determine the general addition laws. The transformations which are to be effected are most simply expressed when the fundamental equations are regarded from a somewhat changed point of view. The general problem is therefore first restated.

Let $\mathscr{E}$ denote the value of the eikonal for a compound system built up of known components. $\mathscr{E}$ is to be regarded simply as the sum of a number of functions, some of which are the eikonals of the components, while others depend upon the separations of the successive reference points. $\mathscr{E}$ is not itself the eikonal for the compound instrument, but merely equal to it in value. It becomes the eikonal itself when the intermediate variables have been eliminated, while the external variables are regarded as constants. The conditions to be applied corresponding to (2) are thus

$$\frac{\partial \mathscr{E}}{\partial M'} = \frac{\partial \mathscr{E}}{\partial N'} = \frac{\partial \mathscr{E}}{\partial M''} = \frac{\partial \mathscr{E}}{\partial N''} = \ldots = 0 \qquad \ldots.(34),$$

and the problem resolves itself into the determination of the value of a function at the point where it is stationary as certain of its variables are changed. The particular feature which renders the problem simple is that the aim in optical instruments is to remove aberrations, or at least reduce them to small amounts, so that all rays are brought nearly into conformity with the paraxial laws. In other words, the problem must be dealt with on the supposition that an approximate solution is given by the terms of the lowest orders.

Elimination of a single variable

For the sake of simplicity consider first the case where only one variable x is to be eliminated from a function $f(x)$ by the application of the condition $f'(x) = 0$. If this condition is expanded by Taylor's theorem it takes the form

$$f' + xf'' + \frac{x^2}{2!}f''' + \frac{x^3}{3!}f'''' + \ldots = 0,$$

where after differentiation x has to be replaced by zero. The terms of the lowest orders give for x the approximate value ξ, where

$$\xi = -\frac{f'}{f''}.$$

The exact value is $$x = \xi - \frac{1}{f''}\phi(x),$$

where $$\phi(x) = \frac{x^2}{2!}f''' + \frac{x^3}{3!}f'''' + \dots,$$

so that
$$\begin{aligned}\phi(\xi) &= \frac{\xi^2}{2!}f''' + \frac{\xi^3}{3!}f'''' + \dots \\ &= f'(\xi) - f' - \xi f'' \\ &= F',\end{aligned}$$

F being written for f when the variable is changed from x to ξ. It follows that the required value of f is given by

$$f = F - \frac{1}{2!f''}(F')^2 + \frac{1}{3!f''^2}\frac{d}{d\xi}(F')^3 - \dots + \frac{1}{(n+1)!(-f'')^n}\frac{d^{n-1}}{d\xi^{n-1}}(F')^{n+1} + \dots \quad \dots(35),$$

where only terms of the second and higher orders are retained in F'. It will be noted that this special case is sufficient for the complete determination of the addition laws for primary aberrations.

Elimination of several variables

Now take the case when n variables $x_1, x_2, x_3, \dots x_n$ are to be eliminated by the application of the conditions

$$f_{x_m} = 0 \quad (m = 1, 2, 3, \dots n),$$

the subscript character denoting differentiation. These conditions may be written

$$0 = f_m + \sum_{i=1}^{n} x_i f_{im} + \dots,$$

where f_m, f_{im} are the values of f_{x_m} and $f_{x_i x_m}$ when all the n variables are made zero. They yield for x_m the approximate value ξ_m, which can be most readily expressed in terms of certain determinants. Denote

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} & \dots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & f_{n3} & \dots & f_{nn} \end{vmatrix}$$

by Δ, and the determinant obtained by replacing the mth row by

$$f_1 \quad f_2 \quad f_3 \quad \dots \quad f_n$$

by Δ_m. Then $$\xi_m = -\frac{\Delta_m}{\Delta}.$$

The exact values for the variables are

$$x_m = \xi_m - \frac{1}{\Delta}\phi_m(x_1, x_2, \dots x_n),$$

and the values of the ϕ's are required when all the x's are replaced by ξ's. It is easy to see that $\phi_m(\xi_1, \xi_2, \dots \xi_n)$ is the value of the determinant obtained by replacing the mth row of Δ by

$$f_{t1} \quad f_{t2} \quad f_{t3} \quad \dots \quad f_{tn}.$$

The stationary value of f with the n variables eliminated may now be written down to any desired degree of approximation by Lagrange's formula. It may be shown that the numerical coefficients are such as to yield a simpler expression for the typical term of the expansion than in the general case. For instance, it will be noticed from the foregoing determinants that if

$$f_{i,j} = 0 \qquad i \neq j \qquad \dots\dots(36),$$

the expressions become

$$\xi_m = -\frac{f_m}{f_{mm}} \qquad \dots\dots(37),$$

and

$$\frac{1}{\Delta}\phi_m(\xi_1, \xi_2, \dots \xi_n) = \frac{f_{tm}}{f_{mm}} \qquad \dots\dots(38),$$

which, as in these circumstances would be expected, are the substitutions corresponding to the elimination of the single variable x_m. Under these conditions, when there are two variables to be eliminated, say x and y, the approximate values of which are ξ and η, the expansion for the stationary value of f assumes the convenient form

$$\begin{aligned} F &+ \tfrac{1}{2}\{aF_\xi^2 + bF_\eta^2\} + \frac{1}{2!}\{a^2F_\xi^2F_{\xi\xi} + 2abF_\xi F_\eta F_{\xi\eta} + b^2F_\eta^2F_{\eta\eta}\} \\ &+ \frac{1}{3!}\left\{a^3\frac{d}{d\xi}(F_\xi^2F_{\xi\xi}) + 3a^2b\frac{d}{d\xi}(F_\xi^2F_\eta F_{\xi\eta}) + 3ab^2\frac{d}{d\eta}(F_\xi F_\eta^2F_{\xi\eta}) + b^3\frac{d}{d\eta}(F_\eta^2F_{\eta\eta})\right\} + \dots \\ &+ \left\{\frac{a^n}{n!}\frac{d^{n-2}}{d\xi^{n-2}}(F_\xi^nF_{\xi\xi}) \right. \\ &\qquad \left. + \sum_{i=1}^{n}\frac{a^{n-i}b^i}{(n-i)!\,i!}\cdot\frac{d^{n-2}}{d\xi^{n-i-1}d\eta^{i-1}}(F_\xi^{n-i}F_\eta^iF_{\xi\eta}) + \frac{b^n}{n!}\frac{d^{n-2}}{d\eta^{n-2}}(F_\eta^nF_{\eta\eta})\right\} + \dots, \end{aligned} \qquad \dots\dots(39),$$

where

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -\frac{1}{a}, & \frac{\partial^2 f}{\partial y^2} &= -\frac{1}{b} \\ \xi &= a\frac{\partial f}{\partial x}, & \eta &= b\frac{\partial f}{\partial y} \end{aligned}\right\} \quad (x = y = 0) \qquad \dots\dots(40),$$

and

$$F = f(\xi, \eta) \qquad \dots\dots(41),$$

the terms used in deriving ξ and η being omitted when F is differentiated.

When the conditions (36) are not satisfied, the final formula assumes a distinctly more cumbersome form than (39). As the investigation stands, it would appear that these desirable conditions are not satisfied; for the differential coefficients which appear in the determinants, though of the lowest orders in the variables which are to be eliminated, involve terms of high orders in the direction cosines of the ray in the extreme media, that is in the ultimate variables of the eikonal for the complete instrument. This consideration also suggests that the actual evaluation of the eikonal will involve much expansion and simplification even after the eliminations have taken place. It is therefore of great importance to enquire into

the conditions more closely and see whether these difficulties can be avoided. For simplicity the case in which only one variable has to be eliminated may be considered. Suppose then that f is a function of $x, z_1, z_2, z_3, \ldots$ and that x is to be eliminated by applying the condition $f_x = 0$. This may be written in the form

$$0 = f_x + xf_{xx} + \Sigma z_i f_{xz_i} + \text{terms of higher orders in } x \text{ and the } z\text{'s},$$

where now the z's as well as x are to be replaced by zero after differentiation. Consider the approximate value ξ for x given by

$$-\xi = \frac{f_x + \Sigma z_i f_{xz_i}}{f_{xx}}.$$

The actual value which satisfies the condition may be written in the form

$$x = \xi - \frac{\phi(x, z_1, z_2, z_3, \ldots)}{f_{xx}},$$

and just as before

$$\phi(\xi, z_1, z_2, z_3, \ldots) = \frac{\partial}{\partial \xi} f(\xi, z_1, z_2, z_3, \ldots).$$

It follows that (35) gives the complete solution to the problem when the additional restrictions are adopted that only the terms of f' which are linear in the ultimate variables are retained, while these variables are altogether excluded from f''. In other words, (35) is the exact solution when ξ is merely the crude first approximation.

Evidently the corresponding result is true in the more general case when several variables are to be eliminated. Under these circumstances the conditions (36) necessarily hold when the system is symmetrical about an axis, and also in the more general case when there is no symmetry if a proper choice of axes is made. It follows that equations (39), (40), and (41) yield in the most convenient form a complete solution of this problem.

The formula is confined to the elimination of two variables because when two systems are amalgamated only two direction cosines have to disappear. When several systems are used in succession some of the intermediate differential coefficients of the form f_{id} are not zero. The expansion can readily be extended to meet this case, but for the purposes of this investigation it appears simpler to extend the results from the particular instance in which two components are involved.

When the component systems are symmetrical about a common axis coinciding with the axis of x, the a and b of equations (40) are equal.

Formulae in terms of G and S

As a first example of the application of (39) consider the development of addition formulae in terms of the variables appropriate for the object and stop magnifications G and S used earlier. The fundamental equation is

$$\mathcal{E} = \eta + e \qquad \ldots\ldots(42),$$

and since the paraxial terms are excluded from the differentiated terms of (39), it at once follows that the addition formulae for the first order aberrations are

given by (26) and (27). To obtain the formulae of the second order, it may be noted that the variables may be supposed to have been changed before differentiation takes place, provided that, when operating on η, it is understood that

$$\frac{\partial}{\partial M'} = -\frac{\phi}{F}\left\{G\frac{\partial}{\partial (M-GM'')} + S\frac{\partial}{\partial (M-SM'')}\right\},$$

and when on e, $\quad \dfrac{\partial}{\partial M'} = \dfrac{f}{F}\left\{\dfrac{\partial}{\partial (M-GM'')} + \dfrac{\partial}{\partial (M-SM'')}\right\}$,

with corresponding meanings for $\dfrac{\partial}{\partial N'}$. Thus

$$F(S-G)^2\,\mathscr{E}_{M'} = (M-GM'')\{-\phi(2G\eta_\kappa + S\eta_\pi) + f(2e_k + e_p)\} + (M-SM'')\{-\phi(G\eta_\pi + 2S\eta_\tau) + f(e_p + 2e_t)\} \quad \ldots.(43),$$

or, utilizing the notation introduced in (29),

$$2(S-G)\,\mathscr{E}_{M'} = (M-GM'')\{(\alpha_0, \alpha_1, \alpha_2 \overline{\wr} K, -2P, T) + \tfrac{1}{2}(S-G)(\alpha_1', \alpha_2' \overline{\wr} P, T)\} - (M-SM'')\{(\alpha_1, \alpha_2, \alpha_3 \overline{\wr} K, -2P, T) + \tfrac{1}{2}(S-G)(\alpha_1', \alpha_2' \overline{\wr} K, P)\}.$$

This leads at once to the relations (30) and (31) for the addition of aberrations of the second order.

In the formulae for the third order aberrations there will occur quantities involving the second order expressions corresponding to the first order expressions in the α's. They may be defined as

$$\beta_i F(S-G) = (\eta_{30i}G - \eta_{30i+1}S)\phi - (e_{30i} - e_{30i+1})f,$$
$$\beta'_{2i-3}F^{-1} = -\eta_{31i}S\phi^{-1} + e_{31i}f^{-1},$$
$$\beta'_{2i-2}F^{-1} = \eta_{31i}G\phi^{-1} - e_{31i}f^{-1},$$

and the terms to be added to the foregoing value of $2(S-G)\,\mathscr{E}_{M'}$ are

$$\frac{(M-GM'')}{4}[3(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \overline{\wr} K, -2P, T)^2 + (S-G)\{P(\beta_1', \beta_3', \beta_5' \overline{\wr} K, -2P, T) + T(\beta_2', \beta_4', \beta_6' \overline{\wr} K, -2P, T) + (\beta_2' + \beta_3')(KT - P^2)\}]$$
$$-\frac{(M-SM'')}{4}[3(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \overline{\wr} K, -2P, T)^2 + (S-G)\{K(\beta_1', \beta_3', \beta_5' \overline{\wr} K, -2P, T) + P(\beta_2', \beta_4', \beta_6' \overline{\wr} K, -2P, T) + (\beta_4' + \beta_5')(KT - P^2)\}].$$

The terms to be retained in $\mathscr{E}_{M'^2}$ and $\mathscr{E}_{N'^2}$ are those which involve both an α and a β.

In addition to these terms are others involving $\mathscr{E}_{M'M''}$, $\mathscr{E}_{M'N'}$, and $\mathscr{E}_{N'N''}$. Now

$$F^2(S-G)^4\,\mathscr{E}_{M'M''} = 2(S-G)^2\{\phi^2(G^2\eta_\kappa + GS\eta_\pi + S^2\eta_\tau) + f^2(e_k + e_p + e_t)\} + \phi^2\left\{(M-GM'')\left(2G\frac{\partial}{\partial\kappa} + S\frac{\partial}{\partial\pi}\right) + (M-SM'')\left(G\frac{\partial}{\partial\pi} + 2S\frac{\partial}{\partial\tau}\right)\right\}^2\eta + f^2\left\{(M-GM'')\left(2\frac{\partial}{\partial k} + \frac{\partial}{\partial p}\right) + (M-SM'')\left(\frac{\partial}{\partial p} + 2\frac{\partial}{\partial t}\right)\right\}^2 e \quad \ldots.(44).$$

In the corresponding expression for $\mathscr{E}_{M'N'}$ the first line of the expression for $\mathscr{E}_{M'M'}$ does not appear, and in each of the other two lines one of the factors involving M's is replaced by the corresponding factor with N's. It follows that

$$\begin{aligned}
&F^2(S-G)^4(\mathscr{E}^2_{M'}\mathscr{E}_{M'M'}+2\mathscr{E}_{M'}\mathscr{E}_{N'}\mathscr{E}_{M'N'}+\mathscr{E}^2_{N'}\mathscr{E}_{N'N'})\\
&\quad=2(S-G)^2(\mathscr{E}^2_{M'}+\mathscr{E}^2_{N'})\{\phi^2(G^2\eta_\kappa+GS\eta_\pi+S^2\eta_\tau)+f^2(e_k+e_p+e_t)\}\\
&\quad+\phi^2\left\{\mathscr{S}\left(2G\frac{\partial}{\partial\kappa}+S\frac{\partial}{\partial\pi}\right)+\mathscr{S}'\left(G\frac{\partial}{\partial\pi}+2S\frac{\partial}{\partial\tau}\right)\right\}^2\eta\\
&\quad+f^2\left\{\mathscr{S}\left(2\frac{\partial}{\partial k}+\frac{\partial}{\partial p}\right)+\mathscr{S}'\left(\frac{\partial}{\partial p}+2\frac{\partial}{\partial t}\right)\right\}^2 e \qquad \ldots.(45),
\end{aligned}$$

where
$$\mathscr{S}=(M-GM'')\mathscr{E}_{M'}+(N-GN'')\mathscr{E}_{N'},$$
$$\mathscr{S}'=(M-SM'')\mathscr{E}_{M'}+(N-SN'')\mathscr{E}_{N'},$$

that is
$$\left.\begin{aligned}
F\mathscr{S}&=K\{-\phi(2G\eta_\kappa+S\eta_\pi)+f(2e_k+e_p)\}\\
&\quad+P\{-\phi(G\eta_\pi+2S\eta_\tau)+f(e_p+2e_t)\}\\
F\mathscr{S}'&=P\{-\phi(2G\eta_\kappa+S\eta_\pi)+f(2e_k+e_p)\}\\
&\quad+T\{-\phi(G\eta_\pi+2S\eta_\tau)+f(e_p+2e_t)\}
\end{aligned}\right\} \qquad \ldots.(46).$$

$\mathscr{S}$ and $\mathscr{S}'$ may thus be expressed in terms of the α's, but the remaining terms of (45) require the introduction of another set of quantities in which the contributions of the first component have ϕ^2 as a factor and are of the second degree in G and S, while those from the second component contain f^2. When these are defined there is no great difficulty in writing down the formulae of the third order, but it is at once apparent that, in this form at any rate, they are not sufficiently concise to commend themselves for normal calculations. The question naturally arises whether the preceding development yields the most convenient expressions for the addition laws.

Simplification of the second order formulae

The addition formulae (26) and (27) relating to the first order aberrations are already in the simplest conceivable form. The leading terms of (30) and (31) are already similarly simple, but the additional terms involving the α's, that is the products of the first order terms, are not as convenient as could be wished, and in (31) in particular the calculations appear lengthy. The desirable direction for simplification to take is the association of the products of terms relating to the first component with the second order coefficients of the same component, the association of the products arising from the second component with the corresponding second order coefficients, and similarly the products of the first order coefficients for the complete lens should be associated with the second order coefficients for the entire system. There will then only remain terms which are incapable of expression in this form, and these will obviously be the products of unlike terms, one from the first, and one from the second, component.

By (39) and (43) the additional terms consist of the product of known terms with the squares and product of

$$-\phi(2G\eta_\kappa+S\eta_\pi)+f(2e_k+e_p)$$

and
$$-\phi(G\eta_\pi+2S\eta_\tau)+f(e_p+2e_t).$$

Consider the square of the first expression. It may be written in the form

$$\begin{aligned}\phi^2(2G\eta_\alpha + S\eta_\pi)^2 + f^2(2e_k + e_p)^2 - 4\phi fG\{(\eta_\alpha + e_k)^2 - \eta_\alpha{}^2 - e_k{}^2\}\\ - 2\phi f(S+G)\{(\eta_\alpha + e_k)(\eta_\pi + e_p) - \eta_\alpha\eta_\pi - e_ke_p\}\\ - \phi fS\{(\eta_\pi + e_p)^2 - \eta_\pi{}^2 - e_p{}^2\} + 2\phi f(S-G)(\eta_\alpha e_p - e_k\eta_\pi),\end{aligned}$$

that is

$$\begin{aligned}\phi fF(S-G)^2\Big[&\frac{2(\eta_\alpha e_p - e_k\eta_\pi)}{F(S-G)}\\ &+\frac{s^2}{\phi(\sigma-\gamma)^2}\{4\gamma\epsilon_\xi{}^2 + 2(\sigma+\gamma)\epsilon_\xi\epsilon_\psi + \sigma\epsilon_\psi{}^2\}\\ &+\frac{1}{f(s-g)^2}\{4ge_k{}^2 + 2(s+g)e_ke_p + se_p{}^2\}\\ &-\frac{1}{F(S-G)^2}\{4G(\eta_\alpha+e_k)^2 + 2(S+G)(\eta_\alpha+e_k)(\eta_\pi+e_p) + S(\eta_\alpha+e_p)^2\}\Big]\end{aligned}$$

....(47),

with corresponding expressions in the other two cases. Combining these gives for the value of the first correcting term in (39),

$$\begin{aligned}&-\frac{K(\eta_\alpha e_p - e_k\eta_\pi) + 2P(\eta_\alpha e_t - e_k\eta_\tau) + T(\eta_\pi e_t - e_p\eta_\tau)}{F(S-G)}\\ &-\frac{\{\xi\epsilon_\xi + \psi\epsilon_\psi + \theta\epsilon_\theta\}\{2\gamma\epsilon_\xi + (\sigma+\gamma)\epsilon_\psi + 2\sigma\epsilon_\theta\} - \frac{1}{2}\{\sigma\xi - (\sigma+\gamma)\psi + \gamma\theta\}\{4\epsilon_\xi\epsilon_\theta - \epsilon_\psi{}^2\}}{\phi(\sigma-\gamma)^2}\\ &-\frac{\{ke_k + pe_p + te_t\}\{2ge_k + (s+g)e_p + 2se_t\} - \frac{1}{2}\{sk - (s+g)p + gt\}\{4e_ke_t - e_p{}^2\}}{f(s-g)^2}\\ &+\frac{\{KE_K + PE_P + TE_T\}\{2GE_K + (S+G)E_P + 2SE_T\} - \frac{1}{2}\{SK - (S+G)P + GT\}\{4E_KE_T - E_P{}^2\}}{F(S-G)^2}\end{aligned}$$

....(48),

where E denotes the direct sum of η and e after the variables have been changed to K, P, T by the paraxial substitution. That is to say, E is the value of $\mathcal{E}$ given by the direct terms only.

The first line of (48) represents the only terms of lower orders which cannot be removed from the expression of the higher order addition laws. The remaining terms are of corresponding form in all three cases, and can therefore be eliminated by assuming a suitable form for the eikonal in general. A convenient way of stating the addition laws is to define the eikonal in terms of another function from which the product terms are absent, and to construct addition formulae for the coefficients of this new function. Thus for instance ϵ may be defined in terms of another function ω, the relation between them, so far as it is derivable from the consideration of the first two terms of (39), being

$$\epsilon = \omega + \frac{\{\xi\omega_\xi + \psi\omega_\psi + \theta\omega_\theta\}\{2\gamma\omega_\xi + (\sigma+\gamma)\omega_\psi + 2\sigma\omega_\theta\} - \frac{1}{2}\{\sigma\xi - (\sigma+\gamma)\psi + \gamma\theta\}\{4\omega_\xi\omega_\theta - \omega_\psi{}^2\}}{\phi(\sigma-\gamma)^2} + \ldots,$$

or more simply

$$\epsilon = \omega - \frac{1}{2\phi}(\omega_M\omega_{M'} + \omega_N\omega_{N'}) + \ldots \qquad \ldots.(49).$$

Let w and W be the corresponding functions associated with e and $\mathscr{E}$ respectively, so that

$$\left.\begin{aligned} e &= w - \frac{1}{2f}(w_{M'}w_{M''} + w_{N'}w_{N''}) + \dots \\ \text{and} \qquad \mathscr{E} &= W - \frac{1}{2F}(W_{M'}W_{M''} + W_{N'}W_{N''}) + \dots \end{aligned}\right\} \quad \dots.(50).$$

Then to the second order of aberrations the relation expressing W in terms of ω and w is

$$W = \omega + w - \frac{K(\omega_x w_y - \omega_y w_k) + 2P(\omega_x w_i - \omega_y w_k) + T(\omega_x w_i - \omega_i w_y)}{F(S-G)} + \dots \quad \dots.(51).$$

A simpler form is

$$W = \omega + w + \frac{1}{2F}(\omega_m w_{m''} + \omega_n w_{n''} - \omega_{m''} w_m - \omega_{n''} w_n) + \dots \quad \dots.(52),$$

where m, n, m'', n'' are written for M, N, M'', N'' to denote that the paraxial substitutions for M' and N' have been made before differentiation takes place. From (51) the simplified second order addition laws may be written down. The expressions for the coefficients of series o are

$$\left.\begin{aligned}
W_{300} &= \omega_{300} + w_{300} - \frac{2}{F(S-G)}(\omega_{200}w_{201} - \omega_{201}w_{200}) \\
W_{301} &= \omega_{301} + w_{301} - \frac{1}{F(S-G)}(\omega_{200}w_{202} - \omega_{202}w_{200}) \\
W_{302} &= \omega_{302} + w_{302} - \frac{2}{5F(S-G)}(\omega_{200}w_{203} - \omega_{203}w_{200} + 2\omega_{201}w_{202} - 2\omega_{202}w_{201}) \\
W_{303} &= \omega_{303} + w_{303} - \frac{1}{10F(S-G)}(\omega_{200}w_{204} - \omega_{204}w_{200} + 8\omega_{201}w_{203} - 8\omega_{203}w_{201}) \\
W_{304} &= \omega_{304} + w_{304} - \frac{2}{5F(S-G)}(\omega_{201}w_{204} - \omega_{204}w_{201} + 2\omega_{202}w_{203} - 2\omega_{203}w_{202}) \\
W_{305} &= \omega_{305} + w_{305} - \frac{1}{F(S-G)}(\omega_{202}w_{204} - \omega_{204}w_{202}) \\
W_{306} &= \omega_{306} + w_{306} - \frac{2}{F(S-G)}(\omega_{203}w_{204} - \omega_{204}w_{203})
\end{aligned}\right\} \quad (53),$$

and for the coefficients of series 1,

$$\left.\begin{aligned}
\frac{W_{312}}{F^2} &= \frac{\omega_{312}}{\phi^2} + \frac{w_{312}}{f^2} - \frac{4(\omega_{200}w_{203} - \omega_{203}w_{200} - 3\omega_{201}w_{202} + 3\omega_{202}w_{201})}{5F^2(S-G)^3} \\
\frac{W_{313}}{F^2} &= \frac{\omega_{313}}{\phi^2} + \frac{w_{313}}{f^2} - \frac{2(\omega_{200}w_{204} - \omega_{204}w_{200} - 2\omega_{201}w_{203} + 2\omega_{203}w_{201})}{5F^2(S-G)^3} \\
\frac{W_{314}}{F^2} &= \frac{\omega_{314}}{\phi^2} + \frac{w_{314}}{f^2} - \frac{4(\omega_{201}w_{204} - \omega_{204}w_{201} - 3\omega_{202}w_{203} + 3\omega_{203}w_{202})}{5F^2(S-G)^3}
\end{aligned}\right\} \quad \dots.(54).$$

The comparison of (53) and (54) with (30) and (31) shows that the changes in the function have effected a marked reduction in the length of the calculations required for numerical computation. It should be particularly noted that the coefficients of the series 1 first order aberrations have disappeared from the second order

formulae. The improvement effected is in fact greater than the appearance of the new formulae suggests, for in some important cases the terms in the brackets disappear. For instance, if the lower order aberrations for the complete instrument are corrected, since the additional terms in (52) may be written

$$-\frac{1}{4F}\{W_m(\omega_{m''}-w_{m''})+W_n(\omega_{n''}-w_{n''})-W_{m''}(\omega_m-w_m)-W_{n''}(\omega_n-w_n)\},$$

only those terms survive which involve the non-aberrational contributions to W. Another illustration is afforded by systems in which η and e, and hence ω and w, are similar apart from dimensional factors.

Third order aberrations

The method already employed for the simplification of the second order aberrations may be applied to (45) to yield the addition formulae for the third order aberrations. Instead of following this course the use of an alternative method will be illustrated.

The paraxial substitution formula is

$$\frac{M'}{F}=\frac{M}{f}+\frac{M''}{\phi} \qquad \ldots.(55),$$

from which it follows that in differentiations on any function of the first component

$$\frac{\partial}{\partial m}=\frac{\partial}{\partial M}+\frac{F}{f}\frac{\partial}{\partial M'}, \qquad \frac{\partial}{\partial m''}=\frac{F}{\phi}\frac{\partial}{\partial M'},$$

and similarly in differentiations on any function of the second component

$$\frac{\partial}{\partial m}=\frac{F}{f}\frac{\partial}{\partial M'}, \qquad \frac{\partial}{\partial m''}=\frac{\partial}{\partial M''}+\frac{F}{\phi}\frac{\partial}{\partial M'}.$$

Differentiations on any function are thus included in the two expressions

$$\frac{\partial}{\partial m}=\frac{\partial}{\partial M}+\frac{F}{f}\frac{\partial}{\partial M'}, \qquad \frac{\partial}{\partial m''}=\frac{\partial}{\partial M''}+\frac{F}{\phi}\frac{\partial}{\partial M'}. \qquad \ldots.(56),$$

with corresponding equations for differentiation with respect to the other direction cosines used as independent variables. If now ω be any function of the direction cosines of the first system, and w be a function of the direction cosines of the second system, (56) gives for differentiations on any function of any or of all these variables

$$\Delta+\delta+d=D+\rho+\tau \qquad \ldots.(57),$$

where

$$\left.\begin{aligned}
\frac{\phi f}{F}\Delta &= (\omega_{M'}+w_{M'})\frac{\partial}{\partial M'}+(\omega_{N'}+w_{N'})\frac{\partial}{\partial N'}\\
2\phi\delta &= \omega_M\frac{\partial}{\partial M'}+\omega_N\frac{\partial}{\partial N'}+\omega_{M'}\frac{\partial}{\partial M}+\omega_{N'}\frac{\partial}{\partial N}\\
2fd &= w_{M'}\frac{\partial}{\partial M''}+w_{N'}\frac{\partial}{\partial N''}+w_{M''}\frac{\partial}{\partial M'}+w_{N''}\frac{\partial}{\partial N'}\\
2FD &= (\omega_m+w_m)\frac{\partial}{\partial m''}+(\omega_n+w_n)\frac{\partial}{\partial n''}+(\omega_{m''}+w_{m''})\frac{\partial}{\partial m}+(\omega_{n''}+w_{n''})\frac{\partial}{\partial n}\\
2F\rho &= \omega_{m''}\frac{\partial}{\partial m}+\omega_{n''}\frac{\partial}{\partial n}-\omega_m\frac{\partial}{\partial m''}-\omega_n\frac{\partial}{\partial n''}\\
2F\tau &= w_m\frac{\partial}{\partial m''}+w_n\frac{\partial}{\partial n''}-w_{m''}\frac{\partial}{\partial m}-w_{n''}\frac{\partial}{\partial n}
\end{aligned}\right\} \quad (58),$$

the differentiations in Δ and D applying to the whole function, those in δ and r to the portions of the function which arise from the first component only, and those in d and ρ to the remainder of the function which arises from the second component. With this understanding (57) operating on $\omega + w$ gives the transformation equivalent to the simplification which has been made in the addition formulae of the second order.

Since (57) is true whatever function is operated on, each of the series of differentiations represented by one side of the equation may be repeated and equated to the repeated operations represented by the other side. This gives

$$\begin{aligned}\Delta^2 + \delta^2 + d^2 + \Delta\delta + \Delta d + \delta\Delta + d\Delta \\ = D^2 + \rho^2 + r^2 + Dp + Dr + \rho D + rD + \rho r + rp \qquad \ldots.(59),\end{aligned}$$

the inversion of the order of successive operations in general changing the result. Let any of these operators of the second order be enclosed in brackets { } to denote that all the differentiations are to be confined to the function operated on at the first stage, the coefficients multiplying the first differential coefficients being regarded as constants during the second differentiations. The order of operators inside these brackets may obviously be changed freely since all the differentiations denoted by them are to be carried out on the same function. When the particular function operated on is $\omega + w$, evidently

$$\Delta^2 = 2\{\Delta^2\}, \quad \delta^2 = 2\{\delta^2\}, \quad d^2 = 2\{d^2\}, \quad D^2 = 2\{D^2\},$$
$$\rho^2 = \{\rho^2\}, \quad r^2 = \{r^2\},$$

and it is not difficult to show that

$$\Delta\delta = \delta\Delta, \quad \Delta d = d\Delta,$$
$$Dp - rD = \{Dp\} - \{rD\},$$
$$Dr - \rho D = \{Dr\} - \{\rho D\}.$$

Also ρ operating on w yields the same result as r operating on ω, and thus for the present case

$$\rho r = \rho^2, \quad rp = r^2.$$

It follows that in the present instance (59) may be written in the special form

$$\{\Delta^2\} + \{\delta^2\} + \{d^2\} + \Delta\delta + \Delta d = \{D^2\} + \{\rho^2\} + \{r^2\} + Dp + Dr \quad \ldots.(60).$$

Now if ∇ be written for Δ when ϵ and e are everywhere substituted for ω and w, the addition theorem (39) takes the form

$$\mathscr{E} = [1 - \tfrac{1}{2}\nabla + \tfrac{1}{2}\{\nabla^2\} - \ldots](\epsilon + e) \qquad \ldots.(61),$$

and if we assume

$$\epsilon = [1 - \tfrac{1}{2}\delta + \tfrac{1}{2}\{\delta^2\} - \ldots]\omega \quad \text{and} \quad e = [1 - \tfrac{1}{2}d + \tfrac{1}{2}\{d^2\} - \ldots]w,$$

equations (57) and (60) show that (61) is equivalent to

$$W = [1 - \tfrac{1}{2}\rho - \tfrac{1}{2}r + \tfrac{1}{2}\{\rho^2\} + \tfrac{1}{2}\{r^2\} - \ldots](\omega + w) \qquad \ldots.(62),$$

provided the relation between $\mathscr{E}$ and W is

$$\mathscr{E} = [1 - \tfrac{1}{2}\mathscr{D} + \tfrac{1}{2}\{\mathscr{D}^2\} - \ldots]W,$$

where $\mathscr{D}$ is D as defined in (58) except that $\omega + w$ is to be replaced by W, that is to say

$$2F\mathscr{D} = W_{m'}\frac{\partial}{\partial m''} + W_{n'}\frac{\partial}{\partial n''} + W_{m''}\frac{\partial}{\partial m'} + W_{n''}\frac{\partial}{\partial n'} \qquad \ldots.(63).$$

It will be observed that $\mathscr{D}$ is exactly the same function for the complete lens that δ and d are for the first and second components respectively.

Equation (62) for W, in conjunction with the definitions of ϵ, e, and $\mathscr{E}$ in terms of ω, w, and W, enables addition formulae to be formed for the third order aberrations, but its appearance suggests that the process of simplification has not been carried to the utmost possible extent. For, since the operators ρ and r are squared, the opposition of sign which was found effective for the reduction of the previous term of the expansion has disappeared, and some at least of the terms included in these repeated operations are likely to be reducible to simpler forms. In considering an alternative form into which to cast these terms it will be observed that they are of the same general type as would be introduced by substituting W for $\omega + w$ in the previous term. We are thus led to consider the effect of operating on W as defined in (62) by ρ and r. The restriction in the scope of these operators hitherto adopted was only required for the purpose of establishing the relation (60), and need be maintained no longer as this end is secured by the convention introduced as to the meaning of operators enclosed in curly brackets.

The relation it is most convenient to employ involves the operators $\rho - r$, that is to say operations of the form

$$(\omega_{m''} + w_{m''}) \frac{\partial}{\partial m} + (\omega_{n''} + w_{n''}) \frac{\partial}{\partial n} - (\omega_m + w_m) \frac{\partial}{\partial m''} - (\omega_n + w_n) \frac{\partial}{\partial n''},$$

and $\rho + r$, which is of the form

$$(\omega_{m''} - w_{m''}) \frac{\partial}{\partial m} + (\omega_{n''} - w_{n''}) \frac{\partial}{\partial n} - (\omega_m - w_m) \frac{\partial}{\partial m''} - (\omega_n - w_n) \frac{\partial}{\partial n''}.$$

Applying the latter to W gives

$$2(\rho + r) W = 2\rho w + 2r\omega - 3\{\rho^2\} w - 3\{r^2\} \omega + \{(\rho - r)^2\} (\omega + w) - \{\rho^2\} \omega - \{r^2\} w + \ldots,$$

and if this relation is used to eliminate the terms in the squared operators of (62) the result is

$$6W - \{(\rho - r)^2\} W = 6\omega - \{\rho^2\} \omega + 6w - \{r^2\} w - (\rho + r)(2W + \omega + w) + \ldots \quad \ldots.(64).$$

In deriving this equation W has been put in place of $\omega + w$ in the second term on the left, as is permissible since the lowest order terms it includes are the highest of those now under consideration. These terms will be removed by assuming in ω, w, and W terms which will neutralize the additions. This is possible since, when operating on a function which includes no terms relating to the second component, a change to the original variables gives

$$\left.\begin{aligned} 2\phi\rho &= \omega_{M'} \frac{\partial}{\partial M} + \omega_{N'} \frac{\partial}{\partial N} - \omega_M \frac{\partial}{\partial M'} - \omega_N \frac{\partial}{\partial N'}, \\ \text{and similarly when no first component terms are present} \\ 2fr &= w_{M'} \frac{\partial}{\partial M''} + w_{N'} \frac{\partial}{\partial N''} - w_{M''} \frac{\partial}{\partial M'} - w_{N''} \frac{\partial}{\partial N'} \end{aligned}\right\} \quad \ldots.(65).$$

The final relations are therefore the definitions of ϵ, e, and $\mathscr{E}$ in terms of ω, w, and W,

$$\left.\begin{aligned}\epsilon &= [1 - \tfrac{1}{2}\delta + \tfrac{1}{2}\{\delta^2\} + \tfrac{1}{8}\{\rho^2\} - \ldots]\,\omega\\ e &= [1 - \tfrac{1}{2}d + \tfrac{1}{2}\{d^2\} + \tfrac{1}{8}\{r^2\} - \ldots]\,w\\ \mathscr{E} &= [1 - \tfrac{1}{2}\mathscr{D} + \tfrac{1}{2}\{\mathscr{D}^2\} + \tfrac{1}{8}\{\mathscr{R}^2\} - \ldots]\,W\end{aligned}\right\} \quad \ldots.(66),$$

where $$2F\mathscr{R} = W_m \frac{\partial}{\partial m'} + W_n \frac{\partial}{\partial n'} - W_{m''} \frac{\partial}{\partial m} - W_{n''} \frac{\partial}{\partial n},$$

and the addition relation

$$W = \omega + w - \tfrac{1}{8}(\rho + r)(2W + \omega + w) + \ldots \quad \ldots.(67).$$

When the variables are changed to K, P, T the addition relation may be written down at once by analogy with (51). For conciseness write

$$\frac{U_{20s}V_{3ij} - V_{20s}U_{3ij}}{F(S-G)} = (bij),$$

where $$U_{xyz} = \frac{\omega_{xyz}}{\phi^{2y}} - \frac{w_{xyz}}{f^{2y}} \text{ and } V_{xyz} = \frac{2W_{xyz}}{F^{2y}} + \frac{\omega_{xyz}}{\phi^{2y}} + \frac{w_{xyz}}{f^{2y}}.$$

Then the third order formulae of series o are

$$\left.\begin{aligned}W_{400} &= \omega_{400} + w_{400} - \tfrac{2}{5}[(001) - (100)]\\ W_{401} &= \omega_{401} + w_{401} - \tfrac{1}{30}[5(002) - 2(101) - 3(200)]\\ W_{402} &= \omega_{402} + w_{402} - \tfrac{1}{70}[10(003) + 5(102) - 12(201) - 3(300)]\\ W_{403} &= \omega_{403} + w_{403} - \tfrac{1}{110}[10(004) + 20(103) - 15(202) - 14(301) - (400)]\\ W_{404} &= \omega_{404} + w_{404} - \tfrac{1}{35}[(005) + 5(104) - 5(302) - (401)]\\ W_{405} &= \omega_{405} + w_{405} - \tfrac{1}{110}[(006) + 14(105) + 15(204) - 20(303) - 10(402)]\\ W_{406} &= \omega_{406} + w_{406} - \tfrac{1}{70}[3(106) + 12(205) - 5(304) - 10(403)]\\ W_{407} &= \omega_{407} + w_{407} - \tfrac{1}{30}[3(206) + 2(305) - 5(404)]\\ W_{408} &= \omega_{408} + w_{408} - \tfrac{2}{5}[(306) - (405)]\end{aligned}\right\} \quad (68),$$

the formulae of series 1 are

$$\left.\begin{aligned}\frac{W_{412}}{F^2} &= \frac{\omega_{412}}{\phi^2} + \frac{w_{412}}{f^2} - \frac{2[(013) - (112)]}{15} - \frac{8[(003) - 3(102) + 3(201) - (300)]}{35F^2(S-G)^2}\\ \frac{W_{413}}{F^2} &= \frac{\omega_{413}}{\phi^2} + \frac{w_{413}}{f^2} - \frac{(014) + 2(113) - 3(212)}{30}\\ &\quad - \frac{2[3(004) - 8(103) + 6(202) - (400)]}{35F^2(S-G)^2}\\ \frac{W_{414}}{F^2} &= \frac{\omega_{414}}{\phi^2} + \frac{w_{414}}{f^2} - \frac{(114) - (312)}{15} - \frac{4[(005) - 2(104) + 2(302) - (401)]}{35F^2(S-G)^2}\\ \frac{W_{415}}{F^2} &= \frac{\omega_{415}}{\phi^2} + \frac{w_{415}}{f^2} - \frac{3(214) - 2(313) - (412)}{30}\\ &\quad - \frac{2[(006) - 6(204) + 8(303) - 3(402)]}{35F^2(S-G)^2}\\ \frac{W_{416}}{F^2} &= \frac{\omega_{416}}{\phi^2} + \frac{w_{416}}{f^2} - \frac{2[(314) - (413)]}{15} - \frac{8[(106) - 3(205) + 3(304) - (403)]}{35F^2(S-G)^2}\end{aligned}\right\} \quad (69),$$

and for the aberration of series 2 the formula is the simple sum

$$\frac{W_{424}}{F^4} = \frac{\omega_{424}}{\phi^4} + \frac{w_{424}}{f^4} \qquad \ldots\ldots(70),$$

as might have been anticipated from its invariant character. The law of formation of the numerical coefficients is very simple, depending on the binomial coefficients for indices 3 and 5.

The conditions for freedom from aberrations

The values which the various coefficients in $\mathscr{E}$ must have for freedom from aberrations to be secured for the magnification G for the image have already been given*, and it has further been shown that important advantages are secured by imposing the additional condition that the centre of the stop is to be free from aberration, particularly in the common case when the stop magnification is unity. As these results stand it is necessary to evaluate the various coefficients of $\mathscr{E}$ from W by (66) or an equivalent equation to see whether the conditions for the removal of the aberrations have been satisfied. To avoid this these conditions will be expressed in terms of the coefficients of W.

Let H and R be new operators in which the coefficients are differential coefficients of $\mathscr{E}$, the definition being

$$\left.\begin{aligned} 2FH &= \mathscr{E}_M \frac{\partial}{\partial M'''} + \mathscr{E}_N \frac{\partial}{\partial N'''} + \mathscr{E}_{M''} \frac{\partial}{\partial M} + \mathscr{E}_{N''} \frac{\partial}{\partial N} \\ 2FR &= \mathscr{E}_M \frac{\partial}{\partial M'''} + \mathscr{E}_N \frac{\partial}{\partial N'''} - \mathscr{E}_{M''} \frac{\partial}{\partial M} - \mathscr{E}_{N''} \frac{\partial}{\partial N} \end{aligned}\right\} \qquad \ldots\ldots(71),$$

so that they correspond exactly to $\mathscr{D}$ and $\mathscr{R}$ with W replaced by $\mathscr{E}$. The solution of equation (66) for the expression of W in terms of $\mathscr{E}$ is

$$W = [1 + \tfrac{1}{2}H + \tfrac{1}{2}\{H^2\} - \tfrac{1}{8}\{R^2\} + \ldots]\,\mathscr{E} \qquad \ldots\ldots(72),$$

that is to say the relation between $\mathscr{E}$ and W is reversible with a change of sign in the operators. Now let $\mathscr{E}$ be the eikonal for a system of lenses free from all aberrations for the magnification G and also free from axial aberration for the magnification S. Then equation (12) of the investigation just mentioned may be written

$$\frac{G}{F(S-G)^2}\mathscr{E} = \tfrac{1}{2}K + \tfrac{1}{8}e_1K^2 - \tfrac{1}{16}(e_2 - 2e_1^2)K^3 + \tfrac{1}{128}(5e_3 - 24e_2e_1 + 24e_1^3)K^4 - \ldots,$$

and the application of (72) at once gives

$$\frac{G}{F(S-G)^2}W = \tfrac{1}{2}K + \tfrac{1}{8}e_1K^2 - \tfrac{1}{16}e_2K^3 + \tfrac{5}{128}e_3K^4 - \ldots. \qquad \ldots\ldots(73).$$

The greater simplicity of the conditions in terms of the coefficients of W would be expected in view of the more complex character of $\mathscr{E}$ exhibited by other results. It will be observed that W contains only the leading terms of $\mathscr{E}$, and that all product terms have disappeared, showing that they correspond to terms of $\mathscr{E}$ which have been eliminated by the substitution of W. There is every reason to expect that

* *O. S.* 23, pp. 315–6.

this result is generally true, in which case the value of W when all aberrations are absent may be expressed in the finite form

$$W = \frac{(S-G)^2}{G} K + (S-G)\left[(1+K)^{\frac{1}{2}} - \frac{1}{SG}(1+S^2K)^{\frac{1}{2}}\right] \quad \ldots.(74),$$

which, apart from the paraxial term, suggests a closer affinity with the characteristic function than with the eikonal.

Further investigations

There are many matters connected with the application of the foregoing results which it has not been possible to include in the present investigation. For example, the invariant relations of the groups of coefficients which are independent of the magnifications of the object and the stop, and as a special case the expression of the coefficients in terms of coefficients for a standard pair of values for these magnifications or alternatively in terms of the coefficients which give the spherical aberration along the axis as already suggested for a reference system. It is hoped to give an account of these matters in a future paper. Before concluding this investigation the form taken by the expressions when more than two systems are to be combined may be noted. The direct terms are of course simply additive. In the product terms the co-factor of any particular lower order coefficient is the difference of the sums of the multiplying coefficients before and after the element considered, the sign of the difference being ascertained from the case of two elements. Thus if the term $\omega_{203} w_{201}$ occurs in the simple case, the coefficient of ω_{203}, where there are n lenses of which the ω refers to the mth, is $-\sum_{1}^{m}(w_{201}) + \sum_{m}^{n}(w_{201})$, the latter sum having the positive sign because in the dual system w relates to the following lens. In the coefficient of ω_{201} the leading coefficient sum would be positive, the following negative.

A NEW PERFECTLY ANALLATIC INTERNAL FOCUSSING TELESCOPE

By E. WILFRED TAYLOR

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ABSTRACT. A new telescope is described which while perfectly anallatic is of the internal focussing type and combines the advantages of the Porro construction with those of the internal focussing telescope.

Introduction

In all surveying instruments, where the distance of an object is deduced from the stadia intercept, the distance so obtained is referred to the "anallatic point," situated either on the axis of the telescope or on the axis of the telescope produced. If the anallatic point does not coincide with the centre of the instrument, a correction must be applied in order to obtain the distance of the object from the centre of the instrument.

Prior to the advent of the internal focussing telescope, the required correction was approximately the sum of the focal length of the object glass and the distance of the object glass from the centre of the instrument.

The introduction of the anallatic lens by Porro automatically referred all distances, deduced from the stadia intercept, to the centre of the instrument, since the latter coincided with the anallatic point.

The internal focussing telescope has now been adopted by most instrument-makers and, as the position of the anallatic point varies, a small but variable correction must be made in order to refer distances to the centre of the instrument.

Telescopes Consisting only of an Object Glass and Eyepiece

If a telescope, of the type in which no internal focussing lens is fitted, is intended for stadia work, the marks in the focal plane of the object glass are usually so spaced that the length of that portion of the levelling staff, the image of which falls between the two marks, is one-hundredth of the distance from the object to the anallatic point of the telescope, a relationship which is true for all distances of the object.

In Fig. 1, B is the object glass, A the anallatic point (situated on the axis of the telescope and at a distance F, equal to the focal length of the object glass in front of the latter), O the centre of the instrument, GH the focal plane of the object glass for infinite rays, gh the focal plane of the object glass for objects at short distance, and x the distance of the object glass from the centre of the instrument.

Let two principal rays, in the plane of the figure, make equal angles with the axis of the telescope and pass through the anallatic point A; after refraction by the object glass B they will emerge parallel and meet the focal planes GH and gh in the points G, H and g, h. The marks G, H and g, h are spaced to correspond to a particular value of the angle α between the principal rays and it will be seen that, as the pencils of rays, two principal rays only of which are shown, must form their image points at positions along the parallel principal rays between the planes GH and gh, the same system of marks will always correspond to the same angle α between the principal rays no matter in what plane the pencils of rays form their image points.

All distances deduced from the stadia intercept are therefore referred to the anallatic point A and can be corrected to the centre of the instrument by adding a linear constant $F + x$. We have assumed the dimensions of x to be constant,

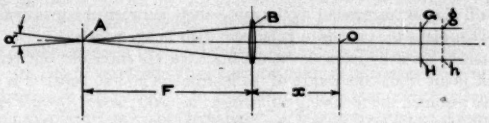

Fig. 1.

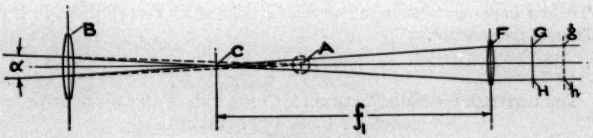

Fig. 2.

which will be the case if the focussing adjustment takes place at the eye end of the telescope. If the focussing adjustment takes place at the object glass end the constant becomes $F + x + y$, where y is the distance through which the object glass is moved from its position when focussed for infinite objects in order to focus on a near object.

The only advantages that can be claimed for this type of telescope are the small loss of light and the small number of glass surfaces to clean.

Porro's Telescope, Consisting of an Object Glass, Anallatic Lens, and Eyepiece

In the Porro telescope all distances deduced from the stadia intercept are automatically referred to the centre of the instrument as the introduction of the anallatic lens causes the anallatic point to coincide with the centre of the instrument.

In Fig. 2, B is the object glass, A the anallatic point situated on the vertical axis of the instrument, F the anallatic lens, GH the focal plane of the combined object glass and anallatic lens for infinite rays, and gh the focal plane for objects at short distance.

Let two principal rays, in the plane of the figure, making equal angles with the axis of the telescope, be directed to the anallatic point A before passing through the object glass; after refraction by the object glass they will meet at the point C (which we will call the secondary anallatic point) on the axis of the telescope and after diverging pass through the anallatic lens F which is situated at a distance f_1 from C equal to its focal length. After passing through the anallatic lens F, which is collective, the principal rays will emerge parallel and meet the focal planes GH and gh in the points G, H and g, h. As before the marks G, H and g, h are spaced to correspond to a particular value of the angle α between the principal rays, and as these rays emerge parallel after passing through the anallatic lens, the same system of marks will always correspond to the same angle between the principal rays, no matter in what plane the pencils of rays form their image points.

All distances deduced from the stadia intercept are therefore referred to both the anallatic point A and the centre of the instrument and no constant is required.

We have assumed the focussing adjustment to take place at the eye end of the telescope, but should it take place at the object glass end, the position of the anallatic point will no longer coincide with the centre of the instrument when the telescope is focussed on near objects and a small plus correction will be theoretically required.

The one important advantage that can be claimed for Porro's telescope is the abolition of the constant.

The Internal Focussing Telescope, Consisting of an Object Glass, Focussing Lens, and Eyepiece

An investigation of the position of the anallatic point is more complicated in the case of the internal focussing telescope, partly owing to changes that take place in the equivalent focal length of the combined object glass and negative lens when focussed for objects at different distances, and partly owing to the changes that take place in the positions of the principal points of the combination. This subject has been thoroughly dealt with by Major E. O. Henrici*.

For distant objects the correction is approximately half the equivalent focal length of the telescope and for very short distances may be considerably less. Fig. 3 is a curve giving the separation between the anallatic point and the centre of the instrument for an 11-inch telescope with the object at different distances. It might be thought that a correction of this magnitude was negligible, but there is still a steady demand for the Porro tacheometer and there are evidently surveyors who appreciate a thoroughly anallatic telescope.

* E. O. Henrici, "The Use of Telescopes with Internal Focussing for Stadia Surveying," *Trans. Opt. Soc.* 22 (1920–21) 20.

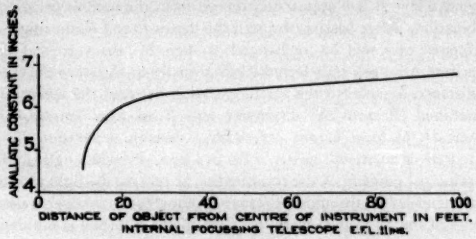

Fig. 3.

A New Internal Focussing Telescope Consisting of an Object Glass, Three Internal Lenses, and an Eyepiece

Before describing a new internal focussing telescope, which is perfectly anallatic, and for which patent application has been made, it is necessary to refer briefly to the combination of lenses shown in Fig. 4, where D is a collective lens of focal length f and E a dispersive lens of focal length $-f$, both situated on the same optical axis with a variable separation K.

The first principal point P_1 of the combination will be at a distance f, equal to its own focal length, from the collective lens D, while the second principal point P_2

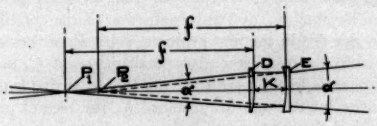

Fig. 4.

of the combination will also be at a distance f, equal to its own focal length, from the dispersive lens E, both principal points being situated on the axis with the collective lens D coming between them and the dispersive lens E.

It will further be apparent from Fig. 4 that if the collective lens D is fixed in position and the dispersive lens E receives a longitudinal movement through the distance K, the first principal point P_1 of the combination will be fixed in position while the second principal point P_2 of the combination will move through a distance K equal to the distance through which the dispersive lens E has been moved; for, being situated at the constant distance f from the dispersive lens E, it necessarily moves with this lens.

The two principal rays, making an angle α with one another, and passing before refraction through the first principal point P_1 of the combination, will after refraction by the collective lens D emerge parallel to the axis and after further refraction

by the dispersive lens E will appear to proceed from the second principal point P_2 of the combination. After passing through the lenses D and E, the angle α between the two principal rays will be unchanged, though P_1, the vertex of the angle α before refraction, has been transferred longitudinally to P_2 after refraction, that is, through a distance K equal to the axial separation between the lenses D and E.

The combined effect of the dispersive lens E, of focal length $-f$, and the collective lens D, of focal length $+f$, with a variable separation K, is exactly equivalent to that of a variable power collective lens, of focal length f^2/K, situated at the first principal point P_1 of the combination to receive the light in question and longitudinally transferred through a distance K (equal to the axial separation between the two lenses D and E) to the second principal point P_2 of the combination to discharge the transmitted light. Such a combination cannot exert collective effect on those rays which proceed to a focus in the principal plane containing the first principal point P_1, but will exert a variable collective effect in the case of any other rays passing through the combination, and converging to or diverging from points in planes other than the principal planes.

Our object now is to apply the optical combination shown in Fig. 4 to the telescope in Fig. 2 in order to obtain all the advantages of an internal focussing telescope, while still retaining the advantage of the Porro telescope, for if by such means we can obtain the effect of a variable power collective lens at the secondary anallatic point C (Fig. 2) where the principal rays cross the axis of the telescope, such an equivalent lens will, as we have shown, cause the pencils of rays to form their image points in different planes along the principal rays and provide a means whereby the image plane can be brought back from the plane gh to the plane GH.

This effect can be obtained by the introduction of the combination shown in Fig. 4. To obtain this result, the first principal point P_1 of the combination must always coincide with the fixed secondary anallatic point C, and therefore the collective lens D must be fixed in position. As the dispersive lens E is capable of a longitudinal movement through the distance K, the second principal point P_2 of the combination will also move through the same distance K and be compensated by a similar longitudinal movement of the anallatic lens F (Figs. 5 and 6), which can be mounted in the same longitudinally adjustable cell as the dispersive lens E with an axial separation S, such that the point P_2 is at a focus of both the lenses E and F.

Referring to Fig. 5, the principal rays proceed in exactly the same manner as we have already described with reference to Fig. 2, until they pass through the collective lens D and the dispersive lens E, of focal length $+f$ and $-f$ respectively. As, however, these two lenses are in close contact, they exercise no refractive effect on the principal rays in question which pass on unchanged in either direction or divergence until they are received by the third lens F, of focal length f_1. As the point C is at the focus of the collective lens F, the principal rays passing through C will emerge from F in a direction parallel to the axis and meet the focal plane GH in two points G and H, where the image points are formed by the pencils of rays from an infinite object, the principal rays of which we have been considering.

If now the telescope is directed to a near object, the pencils of rays will still come to a focus along the same principal rays but in a plane *gh* more remote from the objective *B*. In order to bring the focal plane back to *GH*, in which plane the marks *G* and *H* are situated, the cell, in which the lenses *E* and *F* are mounted with a fixed separation *S*, is moved through a distance *K* (Fig. 6) until the pencils are once more brought to a focus in the plane *GH*.

The principal rays passing through *C* (Fig. 6) will now emerge parallel to the axis after passing through the collective lens *D*, of focal length $+f$, and after passing through the dispersive lens *E*, of focal length $-f$, will diverge as if they came from a point P_2 on the axis, separated from *C* by a distance *K*, which is also the distance through which the lenses *E* and *F* have been moved. Consequently the point P_2 is still at the focus of the collective lens *F*, of focal length f_1, and the

Fig. 5.

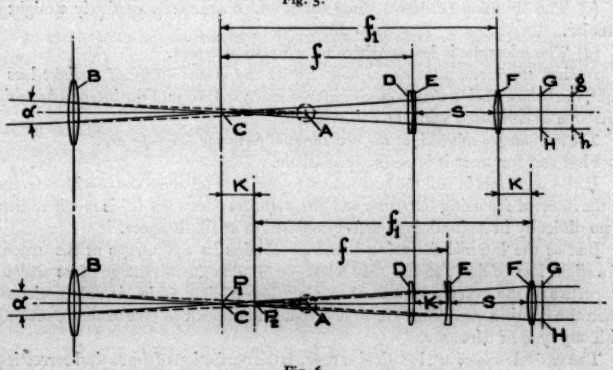

Fig. 6.

principal rays will, after passage through the lens *F*, emerge parallel to the axis and under exactly the same conditions as in Fig. 5 and will meet the focal plane in the same two points *G* and *H*, where the image points corresponding to points in a near object will be formed by the pencils of rays as before.

For all positions of the sliding cell in which the lenses *E* and *F* are mounted between the lens *D* and the focal plane *GH*, the two principal rays, originally directed to the point *A* and making an angle α with one another, will, after passage through the three lenses *D*, *E*, and *F*, emerge in a direction parallel to the axis and pass through the two fixed points *G* and *H*, where by a suitable longitudinal adjustment of the cell the image points will always be formed, provided that the object is within the range of focus of the telescope.

The marks *G*, *H* in the focal plane therefore correspond to the same angle α between the entering principal rays, quite irrespective of the distance of the object,

and since the principal rays are before refraction directed to the anallatic point of the telescope, which coincides with the centre of the instrument, it follows that the telescope is perfectly anallatic for all distances within the compass of the telescope.

It may be of interest to summarize here the advantages generally claimed for the internal focussing telescope, in order to determine what is sacrificed to obtain the perfectly anallatic system in the new instrument.

The advantages claimed for the internal focussing telescope, fitted with a negative lens, are:

(*a*) The fixed and shorter overall length of the telescope enables lower, lighter, and more rigid standards to be used, while still allowing both ends of the telescope to transit at all distances.

(*b*) The telescope can be made air- and water-tight.

(*c*) The abolition of drawn brass tubes which are rarely perfectly straight or circular.

(*d*) The telescope is less sensitive to collimation errors.

Against these real advantages must be set the disadvantage of a slight loss of light, the balance being generally considered much in favour of the internal focussing telescope as compared with the earlier form.

The advantages claimed for the new internal focussing telescope are:

First and foremost it is perfectly anallatic.

It also possesses in full the advantages (*a*), (*b*), and (*c*) given above with reference to the internal focussing telescope and although the advantage (*d*) is modified there is no difficulty in maintaining perfect collimation at all distances.

Just as the internal focussing telescope obtains its advantages at the expense of a slight loss of light, so this new telescope obtains the advantage of its anallatic properties at the expense of a further slight loss of light which is only noticeable when the lighting conditions are so bad that it is difficult to take accurate readings with any type of telescope.

The two telescopes are identical as regards the perfection of the optical corrections and the resolving power.

DISCUSSION

Major E. O. Henrici: The method of obtaining a perfectly anallatic internal focussing telescope is most interesting and is very clearly described. Most surveyors, however, consider that the advantages of the Porro telescope over the simple telescope are not worth the extra complication, weight, cost, etc., involved, and these considerations apply with at least equal force to the telescope described, as compared with the ordinary internal focussing telescope. For many classes of work the additive constant is negligible, especially with the internal focussing telescope, but if the highest accuracy possible is required it must be allowed for, and it is also necessary to determine the multiplying constant under the actual

average conditions of use, as this "constant" depends on atmospheric refraction and consequently varies under different conditions. In such a case this constant is unlikely to be exactly 100, consequently it saves time to construct a table connecting true range with staff intercept, and if this is done it is practically no trouble to allow for the actual additive constant in the table.

Mr T. F. Connolly: The anallatic principle in subtense work has been greatly appreciated for many years. The author combines this with the modern device of an internal focussing lens. The combination has the defects of inevitable extra complexity and cost. For exacting work and where these disadvantages do not apply it should, however, be greatly appreciated.

With the ordinary internal focussing telescope, in spite of some catalogue statements to the contrary, the "constant" is not negligible. In fact, for distances less than 16 ft., with a standard 11-inch instrument the *variation* in the constant becomes of importance.

The author is to be congratulated on an excellent piece of optical design.

Mr T. Smith: In connection with Mr Connolly's remarks, I might point out that it is possible to construct an internal focussing telescope having only two lenses apart from the eyepiece, which will record all distances correctly between suitable working distances to an arbitrarily fixed degree of accuracy, and that the construction may be arranged to make the "constant" zero*.

Mr Taylor is to be congratulated on the system he has designed and so lucidly described. His paper, however, seems to give the impression that the introduction of lenses of opposite sign but numerically equal power is necessary for the construction of such a system. To my mind this does not enter into the problem at all, but the vital factor is the use of a moving telescope for focussing. With four or more lenses there are an indefinite number of solutions to the problem, but for practical purposes it is desirable to have only one portion which moves for focussing. Advantages are secured by having the leading lens and the graticule fixed. The most general arrangement then consists of an optical system divisible into three parts, which in the order in which they are met by the light are

I. A fixed system of lenses.

II. A focussing system which moves as a whole.

III. A fixed system of lenses followed by a fixed graticule.

The third system of lenses may be dismissed at once from further consideration by noting that for this discussion the real graticule may be replaced by its image in this system. The instrument will only give correct values for distances if the constant separation of the stadia lines in the image space corresponds to a constant angle in the object space. In other words the power of the combination I and II must be constant in spite of their varying separation. This requires one or other of them to be telescopic. Again the vertex of the constant angle, which is the first principal focus of the combined system, must be a fixed point of the instrument, and therefore the part which is of finite focal length must be stationary. Thus the

* For details see *Phil. Mag.* 41 (1921) 890.

focussing system II is telescopic, and if it is displaced axially through a distance x the displacement of the second principal focus of the combined systems I and II will be $(1 - 1/m^2)x$, where m is the magnifying power of the focussing telescope.

The anallatic properties of the system have nothing to do with the focussing telescope, but depend entirely on the fixed system I. If a zero correction is required, system I must consist of at least two separated lenses. Let their powers be κ_1 and κ_2, t the separation of their internal principal planes, and c the distance at which the focus is to be behind the first principal plane of the first lens. Then, if the power of system I is A, that of systems I and II combined will be mA, and the construction of I is given by the normal conditions

$$A = (1 + c\kappa_1) = \kappa_2 = (1 + Ac)/t.$$

Mr Taylor: Major Henrici's remarks call for no comment from me, but it is very interesting to know the considered opinion of the Ordnance Survey Department. A steady demand for Porro telescopes still exists, and this seemed to justify an attempt to design an anallatic internal focussing telescope.

My own views are much as expressed by Mr Connolly, and I am interested to know that in his opinion such a telescope may be of value for exacting work.

While admiring Mr Smith's generalized method of attacking the problem, I had personally come to the conclusion that it was impossible to construct an internal focussing telescope, having only two lenses, with a zero constant, though I realized that the constant might be zero for one particular object distance. In Mr Smith's hands the problem appears to have a greater number of solutions than I had thought possible, but I do not entirely follow him when he says that "the anallatic properties of the system have nothing to do with the focussing telescope, but depend entirely on the fixed system." I regret that I have not seen the paper to which he refers, which deals with this problem.

NOTE ON A CONVENIENT BENCH FOR TESTING TELESCOPE OBJECT GLASSES

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MS. received, 14th February, 1924. *Read and discussed, 13th March*, 1924.

ABSTRACT. A description is given of a sloping bench which is suitable for the familiar auto-collimation "star" test of telescope object glasses. Fine mercury globules scattered into black velvet are used to form artificial stars.

Introduction

The bench described in the following note was made to the design of the present writer two or three years ago, and it has proved so useful in the laboratory that a short description may be acceptable to members of the Society.

The test applied is the familiar auto-collimation "star" test in which a good silvered plane mirror is placed behind the lens under examination. The system then forms an image of an artificial star, and the image is nearly coincident with the star object itself, but a slight displacement of the latter enables the image to be examined with a suitably mounted eyepiece or auxiliary microscope. Fig. 1 shows the arrangement. The artificial star is formed by the image of the ball of a "Pointolite" lamp reflected by a small steel ball or a globule of mercury. The lens *L* and the mirror *R* produce an image of the star in the object plane of the microscope *M*. As is well known, the test doubles the apparent spherical and chromatic errors of the lens owing to the double transmission, but a precaution is necessary in using this method to test for coma.

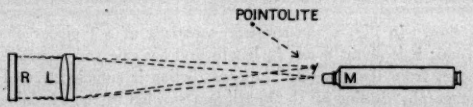

Fig. 1.

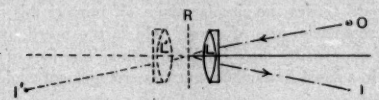

Fig. 2.

In the usual application of the test the objective and eyepiece of the microscope are removed, a "squaring-on" eyepiece* being mounted instead. The lens is then "squared-on," so that it may be assumed that the lens and microscope are co-axial. The mirror *R*, previously covered over, is now uncovered and the artificial star is illuminated. With the aid of the adjusting screws of the mount for the mirror, the star image is then brought into the field of the observing microscope or eyepiece. The necessity for the squaring-on operation is apparent when consideration is given to the possibility shown in Fig. 2. The action of lens *L* and mirror *R* is identical with that of the two lenses *L* and *L'*. In this case the object *O* and image *I* are symmetrically disposed with regard to the axis of the lens, for the reason that the mirror is normal to this axis. The well-known effects of a symmetrical optical system now enter into play, with the result that even if the lens suffers from coma this aberration will not be apparent in the image thus formed. In order, therefore, to examine the extra-axial performance of the lens by the auto-collimating test it is important to secure that when the object is mounted slightly away from the axis the image shall be brought back to the field of the microscope by tilting the mirror, and *not* by displacement of the observing point. The system is then no longer symmetrical and coma will become apparent if it is present. Unless the adjustments are carefully made, it is quite possible that the condition suggested in Fig. 2 may arise, and the absence of apparent coma would not guarantee its absence in the ordinary working of the lens.

The ball for the artificial star can be brought within a few tenths of an inch from the axis of the lens, but in fact the images examined will generally correspond to those which would be found well outside the field of an ordinary eyepiece. Subject to the precautions described above, therefore, the absence of characteristic coma appearances in the auto-collimated image may be taken to indicate the absence of this defect as far as the ordinary employment of the object glass is concerned. Should it be desired to test the images still further from the axis, the most convenient method is to tilt the object glass itself about the nearer nodal point. Arrangements for making this test are included with the bench described below.

The Bench

This is shown in Fig. 3 and consists essentially of a T-girder of stout well-seasoned wood; it is about 8 feet in length and has one end supported at about 4 feet 5 inches from the ground. The construction is sufficiently well illustrated by the photograph. Sliding on the bench is a wooden carriage holding two stout upright boards of deal, each $12'' \times 18''$, the higher being pierced centrally with a hole $4''$ in diameter. Each board has wedged and clamped ends.

The lower board carries the circular plane $4\frac{1}{2}''$ mirror; this rests loosely in a shallow circular receptacle, also of wood, held to the board by a strong spiral spring in tension opposed by three levelling screws. The foot of each screw rests in a conical hole in a small brass plate, but this arrangement could advantageously

* *Trans. Opt. Soc.* 22 (1920–21) 235.

be replaced by the hole, slot, and plane device. Needless to say, the quality of the mirror must be above suspicion.

Fig. 3.

The upper board carries the object glass. This is held in an iris holder on the main wooden mount. The latter (which has its centre removed) rests on three levelling screws passing through the extremities of brass feet screwed to the mount. In this case each screw has its opposing spring, the strength of all the springs being sufficient to prevent the ends of the screws leaving the corresponding holes when the heaviest object glasses which the bench will accommodate are under test.

A simple supplementary device permits the lens to be given a slight rotation about an axis perpendicular to its own optical axis; the lens may also be adjusted with respect to this axis of rotation so that the latter intersects the nodal point. The rotation of the lens is effected by means of a tangent screw with flexible control. In this manner the amount of the rotation can be determined.

At the topmost end of the T-girder is mounted a short piece of triangular optical bench which carries two saddle stands. These support the steel ball or mercury globule and the eyepiece or observing microscope.

The main advantages of the arrangement are found in the simplicity of the mounting and adjustments for lens and mirror; the simple arrangements possible on a sloping bench are not suitable for one which is horizontal. The mirror rests in its support without strain and is in no danger of vibrating, nor is there any fear of its falling. Furthermore the point of observation is at the right height for a person when standing; and the downward direction of view is generally acknowledged to be comfortable.

A small square of black velvet, mounted on cardboard, will retain very fine

mercury globules if these are spattered into the velvet by a sharp tap on a larger drop of mercury. These fine globules, when illuminated by an image of the ball of a "Pointolite" lamp, or some other bright source, make ideal artificial stars; it is possible during observation to get such stars very close to the axis if they lie near the edge of the velvet.

The illumination of the star is a matter for individual experiment, although the mercury globule and "Pointolite" lamp are convenient. The details of the test will doubtless be arranged by each observer to suit his individual needs. It suffices here to call attention to this well-tried method of testing and the apparatus which enables it to be carried out.

My thanks are due to Professor Cheshire for several helpful suggestions in connection with the method of testing and in the details of the construction of the apparatus.

DISCUSSION

Capt. Ainslie: It has occurred to me that the artificial star—or the strip carrying the minute globules of mercury used by Dr Martin—might very conveniently be mounted on the mechanical stage of a microscope, and the image viewed by a low power objective, say, a 2-inch, with a short tube and low power eyepiece. The reflected image could thus be brought readily into the plane of the star, and easily centered so as to be close to it; the artificial star and its image could thus be made very nearly coincident, without the employment of parallel plates set at 45°, which are apt to give double images.

Dr Martin: Capt. Ainslie's suggestion would be very convenient under some conditions, but not, perhaps, when it is desired to perform the test with a simple eyepiece in place of the microscope, as is usually the case in practice.

THE MEASUREMENT OF CHROMATIC ABERRATION ON THE HILGER LENS TESTING INTERFEROMETER

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MS. received, 14th February, 1924. Read and discussed, 13th March, 1924.

ABSTRACT. An account is given of the experimental determination of the longitudinal chromatic aberration and secondary spectrum of telescope object glasses, using the Hilger lens testing interferometer. The amount of mirror displacement necessary to give the same interference pattern for different wave-lengths of light is measured by a screw micrometer.

Introduction

The paper by Mr J. W. Perry*, recently presented to the Society, on the measurement of aberration on the interferometer gave a curve showing the secondary spectrum of a photographic lens; this had been derived by finding the required displacement of the convex mirror of the apparatus when the "best" interference pattern is repeated under illumination with light of various wave-lengths. Within the last few months the present authors, in the course of instructional work, had carried out a few simple experiments on this subject and, in spite of the fact that the first description of the method has now been anticipated, a short account of the work in this laboratory may prove of use in emphasizing the convenience and accuracy of the interferometer for tests of this kind and also in clearing up one or two theoretical points in connection with such tests.

The Apparatus Employed

The work was done on the telescope lens attachment to the ordinary prism testing interferometer, in which collimation of one of the reflected beams is secured by placing a convex mirror of suitable curvature behind the object glass under test. By means of a screw the mirror is displaced until its centre of curvature coincides with the focus of the lens. The axial aberrations are the only ones which can be investigated with this apparatus.

For sources of light a mercury lamp and a hydrogen tube were employed, the required wave-lengths of red, green, blue, and violet light being isolated by suitable colour filters. The colour filters recommended are Wratten filters 62 and 50 for the mercury green and violet lines respectively. Nos. 27 or 29 serve for the hydrogen red line, and any from Nos. 44–49 will give the hydrogen blue-green line in sufficient purity.

* *Trans. Opt. Soc.* 25 (1923–24) 97.

The disposition is shown diagrammatically in Fig. 1. The hydrogen tube *B* is mounted near the pinhole of the interferometer, one end of the capillary portion being placed at the focus of an auxiliary condenser *E*, and the movable mirror *C* is rocked slightly to bring either the mercury light (from the tube *A*) or the hydrogen light into the instrument.

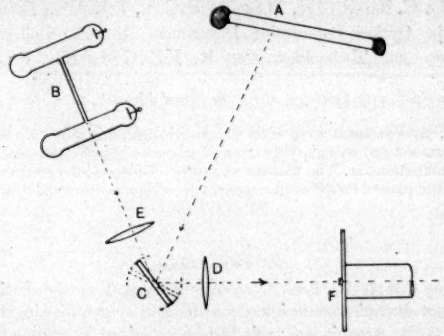

Fig. 1.

In the first simple experiments the displacement of the mirror was found by mounting a circular disc, divided to tenths and twentieths of a revolution, behind the crank of the screw. The handle served as a pointer. The backlash of the screw has to be guarded against. A cardboard disc with a series of holes over which were mounted the (Wratten) colour filters was fitted to the arm carrying the viewing ring of the instrument.

The "best"* interference pattern is found for each available wave-length of monochromatic light and the reading of the micrometer is noted. Three independent settings were made for each colour; it was found possible to set the screw consistently to within 1/20 of a revolution even with these crude arrangements.

Results

The following table gives an example of the experimental results for an achromatic telescope object glass of about 12″ focus and 1½″ aperture.

Colour	Wave-length μ	Micrometer readings 1	2	3	Average	Differences from least reading mm.
Red	·6562	1·14	1·11	1·12	1·12	+0·13
Green	·5461	1·00	0·96	1·00	0·99	0·0
Blue	·4861	1·11	1·12	1·11	1·11	+0·12
Violet	·4359	1·61	1·63	1·64	1·63	+0·64

* For the meaning of this term see below.

The results given in the table are shown in Fig. 2, No. 10. In order to test the method further it was applied to a series of telescope objectives by Messrs Cooke,

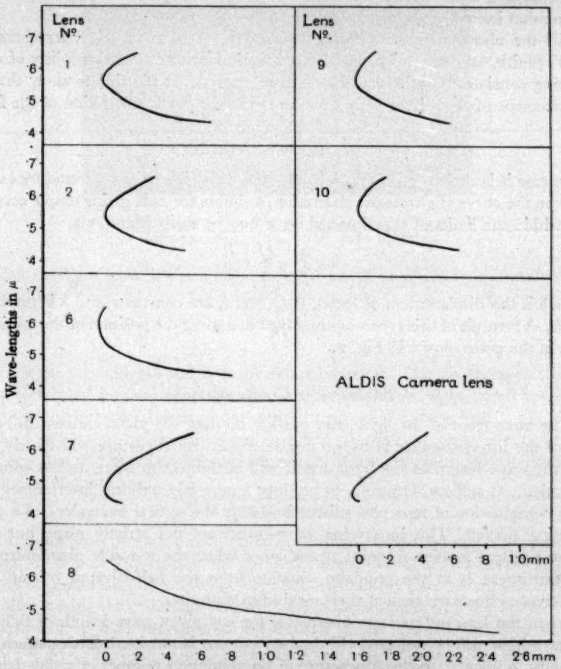

Fig. 2. Displacements of foci of various colours.

Nos. 1, 2, 6, 7, 8, 9, 10 refer to lenses in a "pathological" set of telescope object glasses designed by Messrs Cooke, Troughton and Simms, Ltd., to exhibit errors in construction.

Nos. 1, 2, 9, 10: Colour correction good.

Nos. 6 and 8: Over-correction (moderate and abnormal respectively).

No. 7: Visual under-correction.

Aldis lens: Photographic correction.

Troughton and Simms, Ltd., specially supplied to exhibit various aberrational errors. Some of the resulting curves, and also a curve for an Aldis photographic lens, are shown in Fig. 2. A measurement for the yellow region would have been desirable. This could be well obtained from a helium vacuum tube using the

helium line ·5876μ, but the apparatus was not available at the time of the experiments. The mercury yellow lines are not suitable for the work. No filter will isolate one of them and they give two interference patterns which overlap, thus producing a somewhat indefinite result.

All the above tests have been repeated by Messrs G. H. Carruthers and A. W. Smith, students, with practically identical results. The possibility of easily securing consistent results may be taken as proved. It remains to show that the displacement of the mirror truly represents the chromatic aberration of the focus.

Interpolation Formula

In case it is desired to interpolate between the points actually found by experiment on the curve of chromatic aberration, a simple formula giving ample accuracy well within the limits of experimental error may be easily found, viz.

$$\delta = P + \frac{Q}{\lambda} + \frac{R}{\lambda^2},$$

where δ is the displacement of focus, P, Q, and R are constants, and λ is the wavelength. A formula of this type was employed in finding the position of the minimum focus in the cases shown in Fig. 2.

Theoretical Considerations

The wave-front of the light after passing through the plano-convex collimator lens of the interferometer is by no means plane. Its curvature will clearly vary with the wave-length of the light and it will be subject to more or less spherical aberration. It suffices, however, to produce a perfectly uniform interference field if the combination of lens plus mirror is always the optical equivalent of a plane reflecting surface. The interfering wave-fronts are not strictly plane but their curvatures bear a point to point resemblance when the movable plane mirror of the instrument is at the requisite distance from the half-silvered mirror. The interference effects are then at their maximum visibility.

When the lens suffers from aberration for any given wave-length of light the convex mirror will be adjusted till the best pattern is obtained. This pattern corresponds, in general, to the appearance of the minimum number of visible fringes, but just as it is often difficult to judge the best focus when observing the image given by a lens, so the best interference pattern is not necessarily very easy to select. A similar topic was discussed in a previous paper by one of us*. In the case illustrated in Fig. 3 there is a characteristic pattern due to some astigmatic error in the lens. This comes out most clearly for any wave-length when that part of the fringe system due to the residual symmetrical aberration has been reduced to a minimum by a careful choice of focus.

With short wave-lengths the fringes exhibiting irregularities of figure are necessarily closer together than with long wave-lengths, for, supposing that in

* "A Physical Study of Spherical Aberration," *Trans. Opt. Soc.* 23 (1921-22) 63.

some part of an optical system there is a region in which the thickness of transmitting material in air increases to an excess t over its proper amount in the direction

Fig. 3.
(*a*) green light (*b*) violet light

in which the light travels, the maximum difference δ of phase thus introduced is expressed by

$$\delta = \frac{2\pi}{\lambda}(N-1)t,$$

where λ is the wave-length and N is the refractive index. The photographs (Fig. 3) exhibit clearly the increased number of bands given by violet light, but it is evident that the general characteristics of the contour map will be preserved.

The focus found by the interferometer will be the best physical focus in all cases, and this will take into account the effects (if any) of the chromatic variation of spherical aberration.

Minor Objections

There are one or two minor objections which might be urged against the method. It is well known that the spherical aberration of a lens varies with the distance of the object point. As has already been pointed out, the curvature of the wave after leaving the collimator lens will vary to some extent with the wave-length; this corresponds to an alteration of the distance of the object point. The incident waves are also subject to aberrational distortion varying with the wave-length. It is safe, however, to say from experience that the magnitude of the variation of spherical aberration caused in the test lens by this effect is altogether negligible. The calculation could easily be made if required.

With a lens of very bad figure small errors might at first sight be considered to be caused by the fact that a ray which enters through one zone of the lens emerges through a different zone, and there would be variations of this action with varying wave-length. To demonstrate the negligibility of this effect it suffices to calculate the angular deviation of a part of the interfering wave-front showing the large amount of, say, twenty fringes per centimetre. This corresponds to a retardation of twenty wave-lengths per centimetre or say 10μ per cm. and means an angle of ·001, or somewhat less than 4 minutes of arc. It is at once seen that the lateral

departures of the rays from their proper positions are not sufficient in any ordinary case to cause confusion on this account.

If the plate carrying the half-silvered mirror has been figured to any extent, the correction can only be valid for one wave-length and slight modifications of the true interference pattern for the lens might be obtained with other colours. The magnitude of this effect, however, is doubtless much below any amount which could possibly affect the measurement of chromatic aberration.

Conclusion

The conclusion we reach is that when the necessary apparatus is available the method described above is one of the most accurate and convenient ways of measuring the chromatic aberration of lens systems which can be dealt with on this type of interferometer.

DISCUSSION

Mr J. W. Perry: The interesting results obtained by Dr Martin and Mr Kingslake corroborate the conclusions embodied in the recent paper by the writer on "The Determination of Aberrations*," which, dealing more particularly with the monochromatic aberrations, gave indication also as to how the more obvious procedure of determining the chromatic aberration should be carried out. But the hazy notion of focussing for the "best" focus, which in any intended case involves an uncertain judgment by the observer, appears to be imputed, erroneously, to the writer; whereas the methods described and nomenclature employed (*loc. cit.*) are free from this uncertainty which still characterizes the work of Dr Martin and Mr Kingslake, and which is not proved, simply by the absence of personal inconsistencies, to be without significance. I would suggest that it were better if the chromatic and the spherical aberrations were considered separately, collimation being effected in individual tests specifically for the radiation employed. The authors have, by this paper, made available a very desirable addition to our present knowledge upon this subject in general.

Dr Martin and Mr Kingslake: Mr Perry's paper has not appeared in type at the moment of writing, but we regret that we may have given a wrong impression in referring to his methods. The interesting result from the practical point of view is the chromatic variation of the best physical focus. We select the pattern giving the minimum number of fringes as corresponding to this best physical focus, but we tried to point out that it is not thereby certain that this pattern really corresponds to the focus which would be used in all cases. This is a question at present unsettled and the only "haziness" arises on this account. In designing a lens one has of course to consider spherical and chromatic aberrations separately, but Mr Perry's suggested procedure of a separate collimation for individual wave-lengths was considered needless for the purpose in hand. There is no occasion to try to separate out any spherical aberration effects when measuring chromatic aberration, unless it is desired to check the results of paraxial computations of chromatic error.

* *Trans. Opt. Soc.* 25 (1923–24) 97.

STUDY OF THE SIGNIFICANCE OF THE FOUCAULT KNIFE-EDGE TEST WHEN APPLIED TO REFRACTING SYSTEMS

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[Communicated by Dr L. C. Martin.]

MS. received, 21*st February*, 1924. *Read and discussed*, 13*th March*, 1924.

ABSTRACT. An attempt to explain the various appearances observed in the Foucault knife-edge test led to the construction of accurate three-dimensional models of pencils of light afflicted with aberrations present singly or in pairs. These models are based on equations for the geometrical interpretation of the Seidel aberrations. With their aid the test appearances are explained.

Introduction

The Foucault knife-edge test* is known to be in general use as a means of testing the radii of curvature and the quality of the surfaces of large reflectors. The test seems to be less known as a possible means of testing satisfactorily the performance of comparatively small-aperture refractors. With a view to finding out whether more general use might be made of this simple test a number of experiments were made with refractors having apertures ranging from one-half to three inches.

Apparatus

Most of the experiments to be described were made with lenses selected from the "pathological" set of 1¼-inch telescope objectives supplied by Messrs Cooke, Troughton and Simms, Ltd., of York. These were used chiefly because the aberrations were known for each lens in the set.

The essential apparatus used for the experiments consisted of a short piece of optical bench and a fitting to carry the lenses under test, the means of producing a monochromatic artificial star for use as an object, a good penknife blade, and an eyepiece. The optical bench was not necessary but convenient.

For demonstration purposes, or for taking photographs of the appearances to be described, it was convenient to replace the simple knife blade by the apparatus shown diagrammatically in Fig. 1. This consists of a metal base plate moving along a slot in a short wooden bench, fitted with a millimetre scale. One part of the metal

* See, for references to the subject: Léon Foucault, *Ann. de l'obs. de Paris* (Memoirs) 5 (1859) 200; *Monthly Notices, R.A.S.* (1909) 355; *Phil. Mag.* 48 (1899) 218; *Pogg. Ann.* 131 (1867) 33; *Phil. Mag.* 32 (1916) 409; R. W. Wood's *Physical Optics*, p. 78; *Trans. Opt. Soc.* 24 (1922–23) 207.

base could be clamped to the wooden bench, while the main part carrying the knife-edge K and a vernier V moving over the main scale could be moved small distances

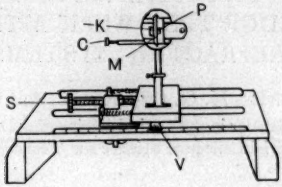


Fig. 1.

by the screw S. The knife-edge could be traversed in its own plane by means of the fine-pitch screw C. The apparatus was originally intended for work on reflectors; for this purpose there was to one side of the knife-edge a small aperture through which light was reflected by the small prism P behind the metal screen M which shielded the observer's eye from light entering the prism.

The camera used for taking photographs of the appearances to be described consisted of a short metal tube into one end of which was placed a small Dallmeyer projection lens; a dark slide held the photographic plate at the other end of the tube.

The artificial star used as an object in the experiments was formed by focussing light from an electric arc with auxiliary condensers and a reversed 1-inch microscope objective into the plane of a small aperture. A green Wratten filter was placed just behind the objective.

Experiments

Each lens to be tested was in turn screwed into the end of a telescope tube, placed on the optical bench fitting, and centered up with respect to the star with the aid of an eyepiece fitted with cross-wires. The knife-edge apparatus was placed in position so that the plane of the knife-edge roughly coincided with the focal plane of the lens. The eye was then placed immediately behind the knife-edge. The position of the knife-edge was found at which the light disappeared most uniformly and quickly from the back surface of the lens when the edge traversed the cone of light normally to the optical axis. If a photograph of the appearance of the back surface of the lens was required, the camera was placed immediately behind the knife-edge, and the photographic image examined on a ground glass screen.

The first experiments were made on lenses having only spherical aberration. Two pairs of lenses were selected, Nos. 1 and 2, known to be good, and Nos. 6 and 10, known to have considerable aberrations. Of the first group, No 2 appeared to be the better; the light vanished uniformly and rapidly as the knife-edge crossed the "best focus." With No. 1, however, it was possible to observe that the disappearance of the light was not quite uniform.

The second pair, containing Nos. 6 and 10, gave more interesting results (see

Fig. 2). In Fig. 2 B the knife-edge has advanced further into the image than in Fig. 2 A. If we consider a geometrical ray diagram for over-correction, the marginal

A *B* *C*

Fig. 2.

A. Lens No. 6 Spherical over-correction. *B*. Lens No. 6 Appearance more developed. *C*. Lens No. 10 Zonal aberration.

focus will be beyond the paraxial focus. If the knife-edge crosses from left to right somewhere between these two foci it will evidently cut out the left marginal rays and the right paraxial rays first of all. This is what has occurred with lens No. 6. A Hartmann test on the lens showed that it was over-corrected spherically.

Now consider the photograph of lens No. 10 given in Fig. 2 C. The knife-edge, approaching the optical axis from the left, has cut out light from the extreme left, an intermediate left zone, from a more or less paraxial zone to the right, and from a wide marginal zone to the right, while a complete ring of light is left at one zone. This is unmistakably zonal aberration. The order of disappearance of the light followed almost exactly the predictions made from consideration of the ray distribution deduced from a Hartmann test made on the lens (see Fig. 3). Supposing

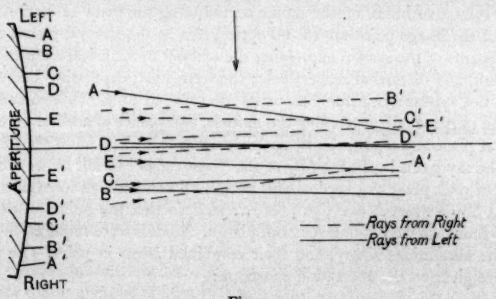


Fig. 3.

the knife-edge to enter the focus in the plane indicated by the arrow, it would cut out light from the back surface of the lens so as to give an appearance very similar to that of Fig. 2 C.

A further series of experiments was made with lenses having coma and astigmatism. The appearances due to spherical aberration were comparatively simple to recognize and to interpret. Much more difficulty was experienced with the oblique aberrations. It was difficult to picture from a plane diagram exactly how the rays passed the focus in an astigmatic or comatic image. Accurate three-dimensional models were therefore made of the ray tracks at and near the foci of cones of light afflicted with one or more types of aberration.

The data for the models were calculated from the following modified forms of the equations deduced by Prof. Conrady* for the geometrical interpretation of the Seidel aberrations:

Horizontal displacement:

$$y = \frac{P}{N'}\left[4a_1 S^3 \cos E + a_2 S^2 V (2 + \cos 2E) + 6a_3 S V^2 \cos E + a_4 V^3\right] + S \cos E \frac{\delta P}{P}.$$

Vertical displacement:

$$z = \frac{P}{N'}\left[4a_1 S^3 \sin E + a_2 S^2 V \sin 2E + 2a_3 S V^2 \sin E\right] + S \sin E \frac{\delta P}{P}.$$

These modified equations take into account the effect of a shift δP of the plane of section along the axis of an oblique pencil from the Petzval plane. For the Petzval plane δP is zero; for any other plane δP will have some positive or negative value. In the equations, P is the axial distance from the exit pupil to the final focus; y and z are the cartesian co-ordinates of the point of penetration of a ray from A in the exit pupil in the plane of section of the cone of rays fixed by the data introduced into the equations; S and angle E are the polar co-ordinates of the point A in the pupil of the system, S being the radius vector (*i.e.* the semi-aperture), and the angle E being counted from the positive direction of the Y-axis towards the point A; N' is the refractive index of the media surrounding the plane of section; V is the distance of the image point above the optical axis in the Petzval plane; a_1, a_2, a_3, a_5 are constants of the system expressing the amount of spherical aberration, coma, astigmatism, and distortion respectively; the term in a_4 expressing Petzval curvature does not appear since the aberrations are referred to the Petzval surface.

For the purpose of constructing the models, two planes of section were selected, the Petzval plane and a plane sufficiently removed from it to allow the intersecting parts of the ray paths in the vicinity of the "best focus" to fall between these two planes. For each plane the vertical and horizontal components were calculated for $S = 1$ and $S = \frac{1}{2}$ when $E = 0°, 30°, 60°, \ldots 360°$, so that the points of penetration of twenty-four rays were found for both planes. Models were required to illustrate each of the aberrations singly, and their combined effect in pairs. The following values were given to the aberration constants:

Sph. ab. model:

$$\frac{P}{N'} \cdot 4a_1 = 1; \qquad a_2 = a_3 = a_5 = 0.$$

* See *Monthly Notices, R.A.S.* (1920) 326.

Coma model:

$$\frac{P}{N'} \cdot a_3 V = \tfrac{1}{3}; \qquad a_1 = a_3 = a_5 = 0.$$

Astig. model:

$$2\frac{P}{N'} \cdot a_3 V^2 = \tfrac{1}{2}; \qquad a_1 = a_2 = a_5 = 0.$$

Sph. ab. + coma model:

$$\frac{P}{N'} \cdot 4a_1 = 1; \qquad \frac{P}{N'} \cdot a_3 V = \tfrac{1}{3}; \quad a_3 = a_5 = 0.$$

Coma + astig. model:

$$\frac{P}{N'} \cdot a_3 V = \tfrac{1}{3}; \quad 2\frac{P}{N'} \cdot a_3 V^2 = \tfrac{1}{2}; \quad a_1 = a_5 = 0.$$

Sph. ab. + astig. model:

$$\frac{P}{N'} \cdot 4a_1 = 1; \quad 2\frac{P}{N'} \cdot a_3 V^2 = \tfrac{1}{2}; \quad a_2 = a_5 = 0.$$

The points thus calculated for each model were plotted on two sheets of cardboard, representing the two planes of section; the planes were suspended parallel to each other and at a convenient separation. Corresponding points in the two planes were then joined by stretched coloured threads, which represented the paths of the rays between the Petzval and the second plane selected.

From these models it was a simple matter to predict the knife-edge test appearances caused by the various aberrations separately or in combination, provided the appearances could be nearly enough explained on geometrical considerations. The predicted appearances of the light distribution at the back surface of a lens as the knife-edge crosses the focus are given in Fig. 4 for pure spherical aberration, coma, astigmatism, and when these are combined in pairs. The arrows indicate the direction of approach of the knife-edge.

Fig. 4 A illustrates spherical aberration (under-correction). This has been dealt with already, but it might be noted here that the appearances are the same for all directions of approach of the knife-edge.

Fig. 4 B gives some of the possible appearances due to astigmatism when the knife-edge enters the cone of light at some position along the optical axis intermediate between the two astigmatic focal lines. Examination of the thread model for astigmatism will show that the appearance varies greatly with the position of the knife-edge along the axis. If the edge happens to cut into the cone of light parallel with and in the plane of one of the focal lines, then the light will disappear uniformly from the lens surface. In any other azimuth this will not be so; the appearance might then be similar to Fig. 4 B (*a*) or (*b*). Lens No. 9 in the pathological set bore out these diagrams very largely, but unfortunately the astigmatism was not pure, and a rather mixed effect was obtained.

Fig. 4 C shows the effects due to coma. The position angles refer to the aperture of the lens under test. By the geometrical interpretation of the Seidel aberrations

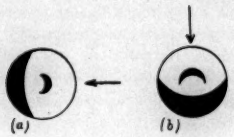


Fig. 4A. Spherical Aberration.

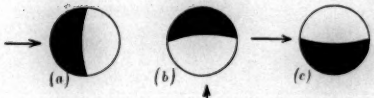


Fig. 4B. Astigmatism.

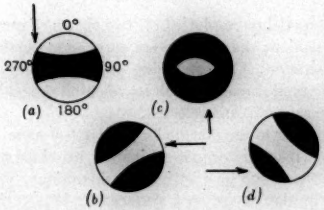


Fig. 4C. Coma.

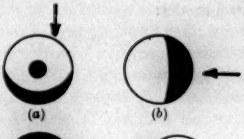


Fig. 4D. Spherical Aberration and Astigmatism.

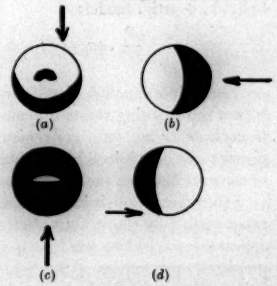


Fig. 4E. Spherical Aberration and Coma.

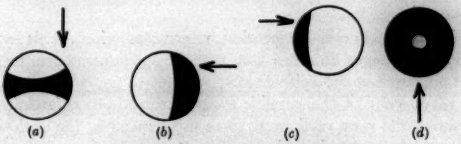
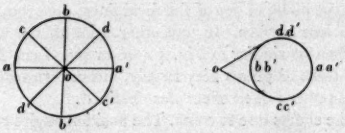


Fig. 4F. Coma and Astigmatism.
N.B. Direction of Coma same in all diagrams.



Aperture Image

Fig. 5.

we know that, in a comatic image, light from points *a*, *a'* (Fig. 5) of the aperture form a focus at *aa'* on the conventional geometric image and that light from *b*, *b'* is focussed at *bb'* in the image, and so on; the paraxial light is focussed at the apex of the geometric coma figure and the point o° in Fig. 4 C therefore corresponds with *a* in Fig. 5. Lens No. 3 of the pathological set almost exactly bore out these predictions indicated for pure coma.

No lenses possessing simple combinations of the aberrations were available for tests. The predicted effects are given to show the importance of trying all directions of approach of the knife-edge before venturing to suggest the defects of the system being tested. Until further trials have been made Fig. 4 D (*a*), for example, might be mistaken for simple spherical aberration; further trials will show that the effects vary with the direction of approach of the knife-edge, and that some of the effects are characteristic of astigmatism. These remarks apply equally well to Fig. 4 E and F.

A last series of experiments was made with a 24 mm. aperture photographic component afflicted with huge amounts of spherical aberration. This lens possessed a series of aperture limiting diaphragms varying from 1 to 22 mm. All data were known for the lens, including the optical path differences for each aperture when working under certain conditions. The lens was carefully set up under these conditions and submitted to the knife-edge test. It was found that the smallest aperture at which the order of disappearance of the light could be detected was such that the path difference was about half the Rayleigh limit. (According to Lord Rayleigh the definition of an optical instrument will not be seriously inferior to that of a perfect instrument if the light arrives at the selected focus with an extreme difference of path not exceeding one-quarter of a wave-length.) This experiment gave one an excellent idea of the appearances due to definite path differences.

Conclusions

After these and a number of unrecorded experiments had been made the following conclusions were arrived at:

(1) Provided the aberrations in the lens system were sufficiently large to cause the light to disappear slowly enough for observation, the chief character of the aberrations and a very fair idea of their quantity could be deduced with a little practice after momentary consideration of the geometric interpretation of the Seidel aberrations. This lower limit at which the aberration can be recognized is below the Rayleigh limit of path difference.

(2) An entirely symmetrical ray distribution cannot produce a symmetrical knife-edge test appearance.

(3) It is of the utmost importance to try the knife-edge in all azimuths before deciding on the types of aberration present. For this reason a good penknife blade is in practice superior to a special piece of apparatus, such as that shown in Fig. 1. A skilled observer can even dispense with the blade and use the edge of the pupil of the eye instead.

(4) The test is of a very similar nature to the star test in that definite quantitative measurements of the aberrations are not made; but, just as in the star test, with a little experience the tester becomes skilled in deciding on the amounts of aberration indicated by the appearances observed.

In conclusion I have to thank Dr L. C. Martin, who suggested the work, for advice and for assistance in referring to the literature on the subject.

DISCUSSION

Capt. Ainslie: I should like to congratulate the authoress of this paper on its interest and importance: the idea of extending the Foucault test to the detection of coma and other asymmetric errors is a distinct step forward. At the same time, the Foucault test always suffers from the disadvantage that it is qualitative rather than quantitative; as used for specula, with an artificial star at the centre of curvature, it tells us whether the figure is on the oblate or prolate side of the sphere, and will indicate when the curve is one of the conic sections, but it does not tell us which. In fact, the method suggested many years ago by Wassell, in which the radii of curvature of different zones of the speculum are measured by the position of the knife-edge when the opposite sides of the zone darken simultaneously, is that more usually adopted.

It may be of interest to note that the use of the pupil of the eye as the knife-edge was suggested by the late Dr A. A. Common, in his description of the construction of his famous 60-inch reflector. I have myself occasionally been able to employ the pupil of the eye in this manner on an actual star image, using the pole star on account of its very slow apparent motion; with a "knife-edge" consisting of a strip of tin-foil across an aperture in the focal plane, and a slow-moving star near the pole, the motion of the star image itself can be made to give the "extinction" effect very conveniently, with a little care in setting the telescope.

Miss Conrady: It should be possible to make the Foucault test quantitative. In the paper it has been pointed out that spherical aberration can be detected within the Rayleigh limit. Anyone frequently using the test would soon learn to recognize the appearance due to a quarter wave-length of aberration. Noting the aperture at which this appears and comparing this with the full aperture of the system, the total amount of aberration could probably be estimated fairly closely. It seems probable that the extra-axial aberrations could also be detected within the Rayleigh limit. Knowing the laws of increase of the primary aberrations with the aperture, the amounts of any primary aberration present at full aperture could then be judged, taking the appearance at the Rayleigh limit as a basis for one's estimations.

REVIEWS

The Physical Society of London: Report on Radiation and the Quantum Theory (Second Edition). By J. H. JEANS. Pp. 86; 3 Figs. (London: Fleetway Press, Ltd., 1924.) Price 7*s*. 6*d*. net; bound in cloth, 10*s*. 6*d*. net.

The appearance of a second edition of this report indeed meets a want only too familiar to those who teach students, for the first sold out very rapidly and each copy must have had many borrowers. The first edition was, at the time, the most admirably written account of the new theory, in a form which made the arguments accessible to a very wide audience whose mathematical equipment was limited. The present volume before us is even more so, and criticism is almost a hopeless task.

One of the outstanding features is its scrupulous fairness, a quality not shared by many writings on this subject. Dr Jeans is not an advocate, and never slurs over or omits to mention a difficulty. If one occurs, he gives a complete account of all that can at present be said on both sides. On such grounds, to the reader who wishes to have a really complete picture of the present state of a rapidly changing and fundamental branch of physics, we have no hesitation in recommending this report as the best account now existing, or likely to exist.

We pass to a brief survey of points of detail. The account of the theory of spectra is naturally entirely changed since the previous edition, which was written so soon after Bohr's original papers. An account of the mode of quantisation due to Wilson and Sommerfeld is perhaps the most significant new feature. But the whole subject of the fine structure of spectrum lines will perhaps appeal as much to the majority of readers—with the emergence of a universal constant not only for spectra themselves in their main laws, but for the fine structure of doublets, triplets, and more complex groupings from all possible elements.

It is unfortunate, though in no sense the fault of the author, that this work should have appeared too soon to include the progress which the quantum theory, in the hands of Curtis and others, is now making towards the elucidation of band spectra. But this adventitious omission is the only one of note which can be pointed out, otherwise the picture is very complete.

The author is somewhat brief on the subject of Bohr's Correspondence Principle, but in this he has our agreement. It is hardly the principle itself which has recently been pushed too far, but the tendency in scientific papers at present is unquestionably to indulge in a somewhat nebulous type of reasoning by analogy, which could be prevented if workers would take the papers of Prof. Bohr himself as their standard. There really is a Correspondence Principle, and, in our judgment, Dr Jeans has deliberately restricted himself to this. But it is a possible criticism which may well be forestalled. In actual fact, it seems probable that the Correspondence Principle serves less, to convince a reader of the essential truth of the quantum theory of spectra, than Ehrenfest's beautiful Principle of Adrabatic Invariance. This, by being very powerful in the elucidation of the implications of the theory, is thereby essentially of a more mathematical type. The author gives an account of it which is very pleasant and convincing, and in which the utmost possible simplification seems to have been made for the more general reader.

The Stark effect is perhaps treated too briefly. The author gives a reference to W. Wilson's more general formulation of the quantum condition, in the addition, for magnetic fields, of a term dependent on the vector potential, and to the further analysis of the Zeeman effect on these lines by Moshanafa. We should have liked to see an actual account of Wilson's modification, but, as in the case of band spectra, the author, pursued by an insistent call for publication, can hardly have had time to incorporate it.

We can only conclude by saying that the work is printed in pleasing type, and is an indispensable one in the library of every physicist.

J. W. N.

Grundzüge der Theorie der optischen Instrumente nach Abbe (Third Edition). By S. Czapski and O. Eppenstein. Pp. xx + 747; 316 Figs. (Leipzig: Johann Ambrosius Barth, 1924.) Price 33 M.

Thirty years ago the article on "Dioptrics and the Theory of Optical Instruments" written by Dr Siegfried Czapski for Winkelmann's *Handbook of Physics* was published as a separate work. Dr Czapski restricted his discussion to such optical instruments as were capable of forming images of external objects and the chief value of his work consisted in the concise generalization of the principles determining the optical design of such instruments. It is largely due to Abbe that such a complete generalization of this particular nature was possible.

In 1904 a second edition was published by the author in collaboration with Dr Eppenstein.

In the third edition, just issued, the original material has been expanded nearly threefold by several of the scientific colleagues of Messrs Carl Zeiss of Jena. This very considerable extension has resulted not only from an elaboration of the original sections, but also from the addition of new ones dealing, for example, with the formation of images by non-spherical surfaces; the action of stops and focal plane screens; diffraction phenomena and their relation to optical instruments; vision as distinct from the section devoted to the eye; spectacle lenses; and also searchlights and image projectors such as epidiascopes. It is inevitable that in a composite work of this kind there should be some repetition, but, apart from the objection that, so far as quantity of material is concerned, the volume is rather unwieldy, such repetition as does occur is hardly a disadvantage, as the treatment in each case is generally characteristic of the particular author.

The first nine chapters of the book are essentially theoretical in character, being devoted to the development of those fundamental expressions that are now so well known to most advanced students of optics. Although far from being technical, the remaining eleven chapters have a more practical appeal.

Presumably the ultimate end of a work of this kind is to facilitate the design of optical instruments. To what extent it fulfils that object must be a matter of individual opinion.

From a perusal of its contents, the uninitiated might conclude that a special mathematical knowledge was the most important part of a designer's equipment. It would be regrettable indeed if any beginner who might seek assistance from this work should form such an opinion and be disheartened. Before he studies a book of this kind, in which what has been previously accomplished is analysed and systematized, he must already be a skilful designer. To acquire such skill a knowledge of general principles—of how to adjust the means available to the end desired and when and to what extent to compromise conditions—is more important in the first instance than any special mathematical accomplishment. It should not be assumed from these remarks that this particular book requires more than an ordinary knowledge of higher mathematics. Where special experience is required is in the filling in of those intermediate steps in the evolution of a mathematical expression that are of necessity so frequently omitted in a work of this kind.

As in the earlier edition, the historical references form a valuable feature of the work, but, to a British reader who may not be able to interpret with ease the German expressions of the various writers, the frequent interruption of the text with so many insertions must be confusing. The contents could be more readily assimilated if these historical references were included separately as foot-notes or confined to an appendix.

James Weir French.